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Sufficient Conditions for Double or Unique Solution of Motion and Structure*

XIAOPING HU AND NARENDRA AHUJA

Beckman Institute and Department of ECE, University of Illinois at Urbana-Champaign, 405 North Mathews Avenue, Urbana, Illinois 61801

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This paper presents several sufficient conditions for a double or unique solution of the problem of motion and structure estimation of a rigid surface from pairs of monocular images. These conditions further the understanding of the uniqueness problem of rigid motion solution. We will show that five correspondences of noncolinear points that do not lie on a special type of quadratic curve, called a Maybank curve, in the image plane suffice to determine a pure rotation uniquely, and six correspondences of points that do not correspond to space points lying on a Maybank quadric suffice to determine a motion with nonzero translation uniquely. We will show that each Maybank quadric can sustain at most two physically acceptable motion solutions and surface interpretations, provided that a sufficient number of correspondences are present. In particular, we will show that in the plane motion case, six correspondences of points that do not lie on a quadratic curve in the image plane will admit only the true motion and structure and their duals as solutions. We will discuss how noise affects the uniqueness of solution and present a nonlinear algorithm for estimation of motion parameters. We will list several properties of the essential matrix $\mathbf{T} \times \mathbf{R}$ and the plane motion matrix $\mathbf{R} + \mathbf{T}\mathbf{N}^T$, both of which are frequently used in the motion and structure estimation problem. Simulation results are provided for verifying the theorems in this paper.

1. INTRODUCTION

This paper concerns the uniqueness of solution of general motion and structure of a rigid surface from two monocular views. This problem can be stated as follows: with how many correspondences of image points and under what conditions can we have a unique solution for $\mathbf{R}$ and a solution up to a scalar for $\mathbf{T}$ from the motion equation

$$Z_i'(x_i', y_i', 1)^T = Z_i\mathbf{R}(x_i, y_i, 1)^T + \mathbf{T}, \quad i = 1, 2, \cdots, n, \tag{1.1}$$

where (x_i', y_i') and (x_i, y_i) are a pair of image point correspondences, $\mathbf{R}$ is a rotation matrix, $\mathbf{T}$ is a vector, and Z_i' and Z_i are any positive constants?

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Many algorithms have been proposed for motion solution (e.g., Tsai [7], Zhuang [6], Weng [19], Longuet-Higgins [13]). However, the only existing condition for the unique solution of general surface motion is having eight correspondences of points that do not lie on a quadric surface passing through the origin of the coordinate system and $-\mathbf{R}_0^T\mathbf{T}_0$, where $\mathbf{R}_0$ and $\mathbf{T}_0$ define the true motion (Longuet-Higgins [14], Zhuang [6]). Although for a planar surface, we have some conditions for determining the motion uniquely (Longuet-Higgins [29], Tsai [8], Weng [18], Hu [26]), so far we do not know if and under what conditions the motion of a plane admits a solution leading to a nonplanar surface.

Longuet-Higgins [28] analyzed the problem of estimating surface structure under the assumption that a sufficiently large number of correspondences are provided. He obtained the following results:

1. If one interpretation of a pair of photographs locates the visible texture elements in a plane, then so does every other.

2. Otherwise, if the images are ambiguous, every interpretation will locate the visible elements on a special type of quadric surface, called a Maybank quadric; but in that case the pair of images cannot sustain more than three distinct and physically acceptable interpretations.

Negahdaripour [12] discussed the relationships of the multiple interpretations of a Maybank quadric and the possible surface shapes of a Maybank quadric.

Since each surface structure must be associated with a motion solution, we then can conclude that a planar surface admits spurious solutions only leading to planes, and a quadric surface admits at most three spurious solutions leading to quadric surfaces. However, these conclusions are made under the assumption that there is a sufficiently large number of correspondences. But how many correspondences are sufficient and what surface conditions they should satisfy to yield a unique or a finite number of solutions are still not known.

The goal of this paper is to present some new and less stringent sufficient conditions for determining motion

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uniquely or to within two sets. We will investigate the problem in several different cases. We will give sufficient conditions for a unique solution of pure rotation, sufficient conditions for unique or double solutions of a general motion with nonzero translation, and sufficient conditions for unique or double solutions of plane motion. Generally speaking, six points satisfying some conditions suffice to determine the motion uniquely. We will show that a Maybank quadric including a planar surface can sustain at most two physically acceptable sets of motion solutions and surface interpretations. We will also present some properties of the essential matrix $\mathbf{T} \times \mathbf{R}$ and the plane motion matrix $\mathbf{R} + \mathbf{T}\mathbf{N}^T$.

We will show that it is feasible to solve for the motion parameters nonlinearly and present an algorithm. Results of simulations will be presented that show that the uniqueness condition for a nonlinear method is generally much less stringent than for a linear method.

When noise is present in the data, the situation is complicated. First, in a strict sense we may not find a rigid motion that is consistent with all of the correspondence data. Therefore we can only find the optimal rigid motion according to some criterion. Then the problem of uniqueness of motion solution is changed to the uniqueness problem of a globally optimal solution of motion under the given criterion. Noise can make a unique solution nonunique and vice versa.

Section 2 introduces the basic notation and some preliminary results on the essential matrix. Section 3 presents some sufficient conditions for the unique solution of a pure rotation. Section 4 presents some sufficient conditions for unique or double solutions of a general motion with a nonzero translation. We will show that a Maybank quadric can sustain at most one other Maybank quadric as a spurious interpretation of the surface. Section 5 investigates the planar surface case and presents some sufficient conditions under which motion solutions other than the true and the dual solutions can be excluded. Section 6 discusses the effect of noise on the uniqueness of the motion solution. Section 7 presents a nonlinear algorithm for motion estimation and some simulation results. Section 8 presents a summary.

2. NOTATION AND PRELIMINARY RESULTS

In this section we will first introduce the motion representation and some basic notation, and then we present some preliminary results which are used in the rest of the paper.

2.1. *The Motion Model and Notation*

We will use the following equation to represent motion:

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \mathbf{R} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \mathbf{T} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}, \tag{2.1}$$

where $\mathbf{R}$ is a rotation matrix, $\mathbf{T}$ is a translation vector, $\mathbf{X} = (X, Y, Z)^T$, and $\mathbf{X}' = (X', Y', Z')^T$ are a pair of corresponding points in space. Without loss of generality, we assume that the projection relation

$$x = \frac{X}{Z}, \quad y = \frac{Y}{Z}, \tag{2.2}$$

holds, where (x, y) is the projection of a point (X, Y, Z) in space on the image plane. We use Θ, Θ' to represent vectors $(x, y, 1)^T$, $(x', y', 1)^T$ so that

$$\mathbf{X} = Z\Theta, \quad \mathbf{X}' = Z'\Theta';$$

we call Θ as well as (x, y) as a point in the image plane. In this paper, all the matrices are 3×3 by default.

Our discussion in this paper will be based on the the so-called *essential matrix* [22],

$$\mathbf{E} \triangleq \mathbf{T} \times \mathbf{R} = \mathbf{G}_T\mathbf{R}, \tag{2.3}$$

where

$$\mathbf{G}_T = \mathbf{T} \times = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1^T \\ \mathbf{g}_2^T \\ \mathbf{g}_3^T \end{bmatrix} \tag{2.4}$$

is a skew-symmetric matrix defined by $\mathbf{T}$. The relation between $\mathbf{G}_T$ and $\mathbf{T}$ is implied by the notation throughout this paper. A matrix is called *essential* or *decomposable* if and only if it is decomposable into the form of (2.3) with $\mathbf{T}$ a nonzero vector and $\mathbf{R}$ a rotation matrix.

2.2. *Preliminary Results*

We will frequently use the following property [24, 26] of a rotation matrix: for any two vectors $\mathbf{T}$ and $\mathbf{X}$,

$$\mathbf{R}(\mathbf{T} \times \mathbf{X}) = (\mathbf{R}\mathbf{T}) \times (\mathbf{R}\mathbf{X}) \tag{2.5}$$

or

$$\mathbf{R}(\mathbf{G}_T\mathbf{X}) = \mathbf{G}_{RT}(\mathbf{R}\mathbf{X}), \tag{2.6}$$

where G_{RT} is a skew symmetric matrix defined by the vector $\mathbf{RT}$. We will call (2.5) as the *rigidity* property, since it is equivalent to the orthonormality and unit determinant property of $\mathbf{R}$ [24, 26]. Equations (2.5) and (2.6) still hold when $\mathbf{X}$ is a matrix.

We first note a duality property of the essential matrix stated in the following lemma.

LEMMA 2.1. *A matrix* $\mathbf{E}$ *can be represented as* $\mathbf{G}_T\mathbf{R}$ *with* $\mathbf{T}$ *a nonzero vector and* $\mathbf{R}$ *a rotation matrix, if and only if it can be represented as* $\mathbf{R}_1\mathbf{G}_{T_1}$ *for some rotation matrix* $\mathbf{R}_1$ *and some nonzero vector* $\mathbf{T}_1$. *Or, a matrix* $\mathbf{E}$ *is essential if and only if there exist two rotation matrices* $\mathbf{R}_i$, $i = 1, 2$, *and a nonzero vector* $\mathbf{T}_3$ *such that*

$$\mathbf{E} = \mathbf{R}_1\mathbf{G}_{T_3}\mathbf{R}_2. \tag{2.7}$$

And if we arbitrarily choose one of $\mathbf{R}_1$, $\mathbf{R}_2$, *and* $\mathbf{T}_3$ ($\mathbf{R}_i$, $i = 1, 2$, *are restricted to be rotation matrices and* $\mathbf{T}_3$ *is a nonzero vector), then we can always find the other two, such that* (2.7) *holds.*

Proof. We first prove the first part of the lemma. We will show that

$$\mathbf{R}[\mathbf{G}_{R^TT}] = \mathbf{G}_T\mathbf{R}. \tag{2.8}$$

From Eq. (2.6) we have

$$\mathbf{R}(\mathbf{G}_{R^TT}\mathbf{R}^T) = \mathbf{G}_{R(R^TT)}(\mathbf{R}\mathbf{R}^T) = \mathbf{G}_T, \tag{2.9}$$

from which Eq. (2.8) follows.

Now let us consider the second part of the lemma. The necessary part is obvious, since for any arbitrary rotation matrices $\mathbf{R}_1$ and $\mathbf{R}_2$ and any nonzero vector $\mathbf{T}_3$, using (2.6) and (2.8) we have

$$\begin{aligned}\mathbf{G}_T\mathbf{R} &= \mathbf{R}_1\mathbf{R}_1^T(\mathbf{G}_T\mathbf{R}) = \mathbf{R}_1\mathbf{G}_{R_1^TT}(\mathbf{R}_1^T\mathbf{R}) \\ &= \mathbf{R}[\mathbf{G}_{R^TT}] = \mathbf{R}[\mathbf{G}_{R^TT}(\mathbf{R}_2^T\mathbf{R}_2)] = (\mathbf{R}\mathbf{R}_2^T)\mathbf{G}_{R_2R^TT}\mathbf{R}_2 \\ &= \mathbf{R}_3^T\mathbf{R}_3(\mathbf{G}_T\mathbf{R}) = \mathbf{R}_3^T[\mathbf{G}_{R_3T}\mathbf{R}_3\mathbf{R}] = \mathbf{R}_3^T\mathbf{G}_{T_3}(\mathbf{R}_3\mathbf{R}),\end{aligned} \tag{2.10}$$

where $\mathbf{R}_3$ is any rotation matrix such that

$$\mathbf{R}_3\mathbf{T} = \mathbf{T}_3. \tag{2.11}$$

The first, second, or last row of (2.10) shows that one of $\mathbf{R}_1$, $\mathbf{R}_2$, or $\mathbf{T}_3$ can be arbitrary (subject to the fact that $\mathbf{R}_1$, $\mathbf{R}_2$ are rotation matrices and $\mathbf{T}_3$ is a nonzero vector) at a time. To prove the sufficient part, we now assume that (2.7) holds. Then according to (2.8) we should have

$$\mathbf{E} = \mathbf{G}_{R_1T_3}(\mathbf{R}_1\mathbf{R}_2), \tag{2.12}$$

where $\mathbf{R}_1\mathbf{T}_3$ is a nonzero vector and $\mathbf{R}_1\mathbf{R}_2$ is a rotation matrix. Therefore $\mathbf{E}$ is an essential matrix. Q.E.D.

We immediately have the following corollary.

COROLLARY 2.1. $\mathbf{E}$ *is decomposable, if and only if* $\mathbf{RE}$ *or* $\mathbf{ER}$ *is decomposable, where* $\mathbf{R}$ *is any rotation matrix;* $\mathbf{E}$ *is decomposable if and only if* $\mathbf{E}^T$ *is decomposable.*

Many necessary and sufficient conditions have been proposed for essential matrices [19, 22–24]; Weng *et al.* and Huang *et al.*'s condition [19, 23], which is in terms of the eigenvalues of $\mathbf{EE}^T$, is the simplest among these (see the Appendix). In this paper, we obtain a condition that is expressed in terms of the elements of $\mathbf{E}$ and may therefore more clearly capture the properties of the essential matrix. We therefore present the following lemma, a proof of which is given in the Appendix.

LEMMA 2.2. *A matrix* $\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]^T$ *can be represented as* $\mathbf{T} \times \mathbf{R}$, *where* $\mathbf{T} = (t_1, t_2, t_3)^T \neq 0$ *and* $\mathbf{R}$ *is a rotation matrix, if and only if there exist three distinct indices* i, j, k *among* 1, 2, 3 *such that one of the following three situations occurs:*

1.

$$\frac{\mathbf{e}_1}{\mathbf{e}_2 \cdot \mathbf{e}_3} + \frac{\mathbf{e}_2}{\mathbf{e}_3 \cdot \mathbf{e}_1} + \frac{\mathbf{e}_3}{\mathbf{e}_1 \cdot \mathbf{e}_2} = 0, \tag{2.13}$$

where

$$\mathbf{e}_m \cdot \mathbf{e}_n \neq 0, \quad \text{for any } m \neq n,\ m, n \in \{1, 2, 3\}. \tag{2.14}$$

In this case,

$$t_1t_2t_3 \neq 0. \tag{2.15}$$

2.

$$\|\mathbf{e}_i\| = \|\mathbf{e}_j\| > 0, \quad \|\mathbf{e}_k\| = 0, \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0. \tag{2.16}$$

In this case,

$$t_i = t_j = 0, \quad \text{but } t_k \neq 0. \tag{2.17}$$

3.

$$\mathbf{e}_i \cdot \mathbf{e}_j = 0, \quad \mathbf{e}_j \times \mathbf{e}_k = 0, \quad \|\mathbf{e}_i\|^2 = \|\mathbf{e}_j\|^2 + \|\mathbf{e}_k\|^2, \quad \|\mathbf{e}_j\| > 0, \quad \|\mathbf{e}_k\| > 0. \tag{2.18}$$

In this case,

$$t_i = 0, \quad t_jt_k \neq 0. \tag{2.19}$$

Longuet-Higgins derived (2.13) (see [23]), which, of course, is one of the conditions for a matrix to be essential.

The next lemma is a generalized version of Zhuang *et al.*'s [6] and will be used in this paper.

LEMMA 2.3. *Assume an arbitrary matrix* $\mathbf{K}$ *is invertible. Then* $\mathbf{K}^T\mathbf{E}$ *is skew-symmetric if and only if* $\mathbf{E} = \mathbf{GK}$ *for some skew-symmetric matrix* $\mathbf{G}$.

Proof. The sufficient part is obvious since $\mathbf{G}$ is skew-symmetric:

$$(\mathbf{K}^T\mathbf{GK})^T = \mathbf{K}^T\mathbf{G}^T(\mathbf{K}^T)^T = -\mathbf{K}^T\mathbf{GK}, \qquad (2.20)$$

which states that $\mathbf{K}^T\mathbf{E}$ is skew-symmetric. Now let us prove the necessary part. If $\mathbf{K}^T\mathbf{E}$ is skew-symmetric, then we should have

$$\mathbf{K}^T\mathbf{E} = -\mathbf{E}^T\mathbf{K}; \qquad (2.21)$$

therefore we have

$$\mathbf{E}^T = -\mathbf{K}^T\mathbf{E}\mathbf{K}^{-1}. \qquad (2.22)$$

Let

$$\mathbf{G} = \mathbf{E}\mathbf{K}^{-1}; \qquad (2.23)$$

then

$$\mathbf{G}^T = (\mathbf{K}^{-1})^T\mathbf{E}^T = -(\mathbf{K}^T)^{-1}\mathbf{K}^T\mathbf{E}\mathbf{K}^{-1} = -\mathbf{E}\mathbf{K}^{-1}. \qquad (2.24)$$

Thus $\mathbf{G}$ is skew-symmetric. It therefore follows from (2.23) that $\mathbf{E} = \mathbf{GK}$ for a skew-symmetric matrix $\mathbf{G}$. Q.E.D.

The following lemma is used in this paper.

LEMMA 2.4. *For a symmetric matrix* $\mathbf{A}$ *to be decomposed into* $\mathbf{E}^T + \mathbf{E}$ *for some essential matrix* $\mathbf{E}$*, the middle eigenvalue of* $\mathbf{A}$ *must be* det($\mathbf{A}$)/2. *Therefore, for a symmetric matrix* $\mathbf{A}$ *to have the form* $\mathbf{E}^T + \mathbf{E}$ *for some essential matrix* $\mathbf{E}$*, there are at most four degrees of freedom in* $\mathbf{A}$ *plus a scale uncertainty.*

Proof. Negahdaripour [12] has proved that if $\mathbf{A}$ has the form $\mathbf{E}^T + \mathbf{E}$ for some essential matrix $\mathbf{E}$, $\mathbf{A}$'s three eigenvalues μ_i, $i = 1, 2, 3$, satisfy the equation

$$\mu_1 + \mu_3 = \mu_2. \qquad (2.25)$$

Since the sum of the three eigenvalues of $\mathbf{A}$ must equal det($\mathbf{A}$), the middle eigenvalue of $\mathbf{A}$ must equal det($\mathbf{A}$)/2. This condition imposes a polynomial equation on the elements of $\mathbf{A}$ and hence restricts at least one degree of freedom of $\mathbf{A}$. For example, when $\mathbf{A}$ is decomposable into $\mathbf{E}^T + \mathbf{E}$, the quadratic curve defined by

$$\mathbf{\Theta}^T\mathbf{A}\mathbf{\Theta} = 0 \qquad (2.26)$$

describes the same shape as the one defined by

$$\mathbf{\Theta}^T \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_1 + \mu_3 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \mathbf{\Theta} = 0. \qquad (2.27)$$

A counterexample is matrix $diag(\alpha, \beta, \gamma)$ with $\beta \neq \alpha + \gamma$ which cannot be decomposed into $\mathbf{E}^T + \mathbf{E}$ for any essential matrix $\mathbf{E}$. Since a symmetric matrix has only five free variables plus a scale uncertainty and since changing the scale does not change the decomposability of a symmetric matrix, we have thus proved the lemma. Q.E.D.

To simplify the statements, we will call a curve of the type (2.26), with $\mathbf{A}$ an essential matrix, a *Maybank curve* from now on.

Now let us reformulate the uniqueness problem which we stated in Section 1 for the sake of obtaining sufficient conditions. Let $\mathbf{R}_0$ and $\mathbf{T}_0$ denote the true motion parameters. The nonuniqueness of the motion solution is equivalent to having a rotation matrix $\mathbf{R}$ and a translation vector $\mathbf{T}$ such that (1.1) and one of the following equations hold:

$$\mathbf{R} \neq \mathbf{R}_0 \quad \text{or} \quad \mathbf{T} \times \mathbf{T}_0 \neq 0 \quad \text{or both (when } \mathbf{T}_0 \neq 0) \qquad (2.28)$$

or

$$\mathbf{R} \neq \mathbf{R}_0 \quad \text{or} \quad \mathbf{T} \neq 0 \quad \text{or both (when } \mathbf{T}_0 = 0). \qquad (2.29)$$

When a spurious solution for $\mathbf{T}$ and $\mathbf{R}$ exists, we must have the following *motion epipolar line equation*

$$(x_i', y_i', 1)(\mathbf{T} \times \mathbf{R})(x_i, y_i, 1)^T = 0. \qquad (2.30)$$

The above equation can always be deduced from (1.1) but the inverse is not true. It is easy to show that when $\mathbf{T} \neq 0$ and (2.30) is satisfied then we can always find two numbers Z_i' and Z_i (not necessarily positive), which are not simultaneously zero, such that (1.1) holds. By multiplying (2.30) with the true depths Z_i' and Z_i we obtain

$$\mathbf{X}_i'^T(\mathbf{T} \times \mathbf{R})\mathbf{X}_i = 0, \quad i = 1, 2, \cdots, n. \qquad (2.31)$$

Equation (2.30) constitutes a necessary condition for the existence of a spurious solution. Therefore, if we can find a condition under which (2.30) or (2.31) does not hold for any spurious solution, then that condition is a sufficient condition for the unique solution of motion parameters.

Further, if for all $\mathbf{R}$ and $\mathbf{T}$ satisfying (2.30) or (2.31), the computed depths Z_i' and Z_i from (1.1) are not all positive simultaneously, then the motion is also uniquely determined by the correspondence data.

With these preliminary results, we can focus now on the main goal of this paper, which is to show that (2.31) can be expressed in terms of surface conditions.

3. SUFFICIENT CONDITIONS FOR DETERMINING A PURE ROTATION

In this section we present some sufficient conditions under which a unique motion solution is guaranteed when the motion is a pure rotation. We assume that the true motion is a pure rotation in this section, but we do not assume that we know this condition in advance. If the motion is a pure rotation, then two point correspondences suffice to determine the rotation uniquely [24], as stated in the following theorem.

THEOREM 3.1. *If the motion is a pure rotation* $\mathbf{R}_0$, *then two correspondences of image points* $\mathbf{\Theta}_i = (x_i, y_i, 1)^T$, $\mathbf{\Theta}_i' = (x_i', y_i', 1)^T$, $i = 1, 2$, *exclude any spurious pure rotation solutions, and the rotation matrix is uniquely given by*

$$\mathbf{R} = [\gamma_1\mathbf{\Theta}_1', \gamma_2\mathbf{\Theta}_2', (\gamma_1\mathbf{\Theta}_1') \times (\gamma_2\mathbf{\Theta}_2')][\mathbf{\Theta}_1, \mathbf{\Theta}_2, \mathbf{\Theta}_1 \times \mathbf{\Theta}_2]^{-1}. \quad (3.1)$$

where

$$\gamma_i \triangleq \frac{Z_i'}{Z_i} = \frac{\|\mathbf{\Theta}_i\|}{\|\mathbf{\Theta}_i'\|} = \sqrt{(x_i^2 + y_i^2 + 1)/(x_i'^2 + y_i'^2 + 1)}, \quad i = 1, 2. \quad (3.2)$$

To solve for $\mathbf{R}$ with the constraint that $\mathbf{R}$ is a rotation matrix it is necessary and sufficient to have

$$\gamma_1\mathbf{\Theta}_1' \cdot \gamma_2\mathbf{\Theta}_2' = \mathbf{\Theta}_1 \cdot \mathbf{\Theta}_2, \quad (3.3)$$

as when (3.3) is satisfied and γ_i, $i = 1, 2$, are defined by (3.2), the matrix on the right-hand side of (3.1) is orthonormal and of unit determinant. This can be readily shown by proving that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$ with $\mathbf{R}$ represented by the right-hand side of (3.1).

However, this solution can be used only after we have already known that the motion is a pure rotation. To make sure the motion is a pure rotation and no other motion solutions will produce the same projections of points in the image plane, we need more point correspondences satisfying some conditions. The next theorem deals with this problem.

THEOREM 3.2. *If the motion is a pure rotation, then a sufficient condition for determining the motion uniquely is having five correspondences of noncolinear points that do not lie on a Maybank curve in the image plane.*

Proof. We need only to consider the case where the spurious solution has a nonzero translation. The case where the spurious solution is a pure rotation has been discussed in Theorem 3.1.

Assume $\mathbf{R}_0$ is the true rotation; then for any image point correspondence pair $\mathbf{\Theta}'$ and $\mathbf{\Theta}$ we should have

$$\gamma_i\mathbf{\Theta}_i' = \mathbf{R}_0\mathbf{\Theta}_i \quad \text{for any } i, \quad (3.4)$$

where γ_i is defined in (3.2). Now if there is another motion solution $\mathbf{R}$ and $\mathbf{T}$ with $\mathbf{T}$ a nonzero translation vector such that (1.1) is satisfied, then the motion epipolar line equation must also be satisfied, i.e.,

$$\mathbf{\Theta}_i'^T(\mathbf{G}_T\mathbf{R})\mathbf{\Theta}_i = 0. \quad (3.5)$$

Substituting (3.4) for $\mathbf{\Theta}_i'$ into (3.5) leads to

$$\mathbf{\Theta}_i^T[\mathbf{R}_0^T(\mathbf{G}_T\mathbf{R})]\mathbf{\Theta}_i = 0. \quad (3.6)$$

The above equation indicates that the correspondence points must lie on a quadratic curve in the image plane unless $\mathbf{R}_0^T\mathbf{G}_T\mathbf{R}$ is a skew-symmetric matrix which makes (3.6) trivial. We now show that the case that $\mathbf{R}_0^T\mathbf{G}_T\mathbf{R}$ is skew-symmetric can be excluded with three noncolinear point correspondences.

According to Lemma 2.3, $\mathbf{R}_0^T\mathbf{G}_T\mathbf{R}$ is skew-symmetric if and only if

$$\mathbf{G}_T\mathbf{R} = \mathbf{G}_{T_1}\mathbf{R}_0 \quad (3.7)$$

for some vector $\mathbf{T}_1$. Since an essential matrix can have only two representations of the form $\mathbf{G}_T\mathbf{R}$ and if $\mathbf{G}_T\mathbf{R}$ is a representation then $-\mathbf{G}_T\mathbf{R}_1$ for some rotation matrix $\mathbf{R}_1$ is also a representation [7, 11], (3.7) implies that

$$\mathbf{T}_1 = \pm\mathbf{T}. \quad (3.8)$$

First assume that $\mathbf{T}_1 = \mathbf{T}$; then $\mathbf{R} = \mathbf{R}_0$. Then from the assumption that $\mathbf{T}$ and $\mathbf{R}$ constitute a motion solution for the correspondences, we have

$$Z_i'\mathbf{\Theta}_i' = Z_i\mathbf{R}_0\mathbf{\Theta}_i + \mathbf{T} \quad \text{for any } i, \quad (3.9)$$

where Z_i' and Z_i are two positive numbers. Substituting (3.4) into (3.9) leads to

$$(Z_i' - \gamma_iZ_i)\mathbf{\Theta}_i' - \mathbf{T} = 0 \quad \text{for any } i. \quad (3.10)$$

By assumption that $\mathbf{T} \neq 0$, we must have

$$\mathbf{\Theta}_i' \times \mathbf{T} = 0 \quad \text{for any } i, \quad (3.11)$$

from which we can conclude that $\mathbf{T} = 0$ with three correspondences that are not colinear in the image plane, contradicting the assumption that $\mathbf{T} \neq 0$.

Now assume that $\mathbf{T}_1 = -\mathbf{T}$. Then (3.7) yields

$$\mathbf{T} \times (\mathbf{R} + \mathbf{R}_0) = 0. \tag{3.12}$$

Again, from the assumption that $\mathbf{T}$ and $\mathbf{R}$ constitute a motion solution for the correspondences, we have

$$Z_i'\Theta_i' = Z_i\mathbf{R}\Theta_i + \mathbf{T} \quad \text{for any } i. \tag{3.13}$$

Multiplying both sides of (3.4) by Z_i and then adding the resulting equation to (3.13), we obtain

$$(Z_i\gamma_i + Z_i')\Theta_i' = Z_i(\mathbf{R}_0 + \mathbf{R})\Theta_i + \mathbf{T}. \tag{3.14}$$

Using $\mathbf{T}$ to cross multiply (3.14) and applying (3.12) to the resulting equation we obtain

$$(Z_i\gamma_i + Z_i')\mathbf{T} \times \Theta_i' = 0 \quad \text{for any } i. \tag{3.15}$$

Because $Z_i\gamma_i + Z_i'$ is a positive number, we then must have (3.11). Again we can conclude that $\mathbf{T} = 0$ with three correspondences that are not colinear in the image plane.

Therefore, given three or more correspondences of noncolinear points, if they admit a solution with nonzero translation, the corresponding points must lie on a quadratic curve defined by (3.6), where $\mathbf{R}_0^\tau(\mathbf{G}_T\mathbf{R})$ is an essential matrix. Since Lemma 2.1 shows that any essential matrix can be put into the form of $\mathbf{R}_0^\tau(\mathbf{G}_T\mathbf{R})$ for some $\mathbf{T}$ and $\mathbf{R}$, therefore, any Maybank curve of the form (2.26) would satisfy (3.6). Now assume that $\mathbf{A}$ is symmetric; then if and only if $\mathbf{A}$ can be decomposed into $\mathbf{E} + \mathbf{E}^\tau$ for some essential matrix $\mathbf{E}$, the curve equation (2.26) can be reduced to (3.6). Lemma 2.4 states that $\mathbf{A}$ has at most four degrees of freedom. Therefore, given five points, it is possible that there exists no Maybank curve passing through all five points. It is therefore sufficient to have five correspondences of noncolinear points that do not lie on a Maybank curve in the image plane to exclude any spurious solution with nonzero translation. Q.E.D.

The condition in the above theorem is generally sufficient but not necessary. It may be possible to determine a pure rotation from four correspondences only (see the example in Section 7), although a sufficient condition for determining a pure rotation with four correspondences is still not available.

4. SUFFICIENT CONDITIONS FOR DETERMINING A MOTION WITH NONZERO TRANSLATION

In this section, we consider general motion where $\mathbf{T} \neq 0$. The situation in this case is much more complicated than when $\mathbf{T} = 0$.

There exists a necessary and sufficient condition which states that if and only if there are eight or more correspondences of points that do not lie on a quadric surface passing through the origins of the coordinate system before and after motion then a motion with a nonzero translation can be determined linearly and uniquely (Longuet-Higgins [14], Maybank [27]). Thus, *any* quadric surface passing through the origin before and after motion will make the linear solution degenerate.

But how many correspondences are needed and what conditions the points should satisfy to uniquely determine the motion *nonlinearly* are still not known. We now present some sufficient conditions for unique solution of motion parameters, which are less stringent than the above stated necessary and sufficient condition for the linear algorithm.

First we have the following theorem which gives a sufficient condition for excluding a pure rotation as a solution and it is quite similar to Theorem 3.2.

THEOREM 4.1. *If the true motion involves a rotation* $\mathbf{R}_0$ *and a nonzero translation* $\mathbf{T}_0$, *a sufficient condition for excluding a pure rotation as a solution is having five correspondences of noncolinear points that do not lie on a Maybank curve in the image plane.*

Proof. Let $\Theta' = (x', y', 1)^\tau$ and $\Theta = (x, y, 1)^\tau$ be a correspondence pair in the image plane. Then the motion epipolar line equation must hold, i.e., we should have

$$\Theta'^\tau(\mathbf{T}_0 \times \mathbf{R}_0)\Theta = 0. \tag{4.1}$$

Now assume a pure rotation $\mathbf{R}$ yields the same correspondence pair, then we should also have

$$\gamma\Theta' = \mathbf{R}\Theta, \tag{4.2}$$

where γ is similarly defined as γ_i is in (3.2). Substituting (4.2) into (4.1) gives

$$\Theta^\tau\mathbf{R}^\tau(\mathbf{T}_0 \times \mathbf{R}_0)\Theta = 0. \tag{4.3}$$

Equation (4.3) indicates that the points must lie on a Maybank curve. As in the proof of Theorem 3.2, we can then conclude that a sufficient condition for excluding a pure rotation as a solution is having five correspondences of noncolinear points that do not lie on a Maybank curve in the image plane. Q.E.D.

But how many point correspondences are needed and what condition should they satisfy to exclude any spurious solution with nonzero translation? The theorem below deals with this problem. However, before we present the theorem, we consider two particular spurious solutions: in one only the translation is distinct; in the other only the rotation matrix is distinct.

First, let us consider the situation where only the translation is distinct.

LEMMA 4.1. *If the true motion is given by* $\mathbf{R}_0$ *and* $\mathbf{T}_0 \neq 0$, *then three correspondences of points that are non-colinear in the image plane will be sufficient for excluding any spurious solution with the rotation* $\mathbf{R}_0$ *and a translation* $\mathbf{T}$ *not parallel to* $\mathbf{T}_0$.

Proof. For a given rotation $\mathbf{R}_0$, the translation $\mathbf{T}_0$ can be uniquely determined with two correspondences Θ_i, Θ_i', $i = 1, 2$, from

$$(\Theta_i' \times \mathbf{R}_0\Theta_i) \cdot \mathbf{T}_0 = 0, \quad i = 1, 2, \tag{4.4}$$

if and only if the four points Θ_i' and $\mathbf{R}_0\Theta_i$, $i = 1, 2$, are not colinear in the image plane. Therefore three correspondences of points that are non-colinear in the image plane will be sufficient for excluding any spurious solution with the rotation $\mathbf{R}_0$ and a translation $\mathbf{T}$ not parallel to $\mathbf{T}_0$. Q.E.D.

We now consider the situation where only the rotation matrix is distinct.

LEMMA 4.2. *If the true motion is given by* $\mathbf{R}_0$ *and* $\mathbf{T}_0$, *a sufficient condition for excluding any spurious solution with a rotation* $\mathbf{R} \neq \mathbf{R}_0$ *and a translation* $\mathbf{T}$ *parallel to* $\mathbf{T}_0$, *is having five correspondences of points that do not lie on a Maybank curve in the image plane.*

Proof. Given a space point correspondence $\mathbf{X}' = Z'\Theta'$ and $\mathbf{X} = Z\Theta$, we have

$$\mathbf{X}' = \mathbf{R}_0\mathbf{X} + \mathbf{T}_0. \tag{4.5}$$

Although we only know Θ' and Θ, Z' and Z are uniquely determined by (4.5) unless Θ' is parallel to $\mathbf{T}_0$. In any case, for the true depths Z' and Z, Eq. (4.5) is always satisfied. Now assume we have a spurious motion solution $\mathbf{R}$ and $\mathbf{T} \neq 0$. Then the motion equipolar line equation

$$\Theta'^T\mathbf{E}\Theta = 0 \tag{4.6}$$

must be satisfied, where $\mathbf{E} = \mathbf{G}_T\mathbf{R}$. Multiplying the left side of (4.6) by $Z'Z$ yields

$$\mathbf{X}'^T\mathbf{E}\mathbf{X} = 0. \tag{4.7}$$

Substituting (4.5) into (4.7) we obtain

$$\mathbf{X}^T(\mathbf{R}_0^T\mathbf{E})\mathbf{X} + \mathbf{T}_0^T\mathbf{E}\mathbf{X} = 0. \tag{4.8}$$

Equation (4.8) is called *Maybank quadric* [28]. With $\mathbf{T} \times \mathbf{T}_0 = 0$, (4.8) reduces to a Maybank curve:

$$\mathbf{X}^T(\mathbf{R}_0^T\mathbf{E})\mathbf{X} = \Theta^T(\mathbf{R}_0^T\mathbf{G}_T\mathbf{R})\Theta = 0. \tag{4.9}$$

According to the results in the proof of Theorem 3.2, five correspondences of image points that do not lie on a Maybank curve suffice to invalidate (4.9) and, hence, exclude a spurious solution with a translation vector parallel to the true translation. Q.E.D.

Finally, let us consider the situation where $\mathbf{R} \neq \mathbf{R}_0$ and $\mathbf{T} \times \mathbf{T}_0 \neq 0$. Let us rearrange (4.8) as

$$\mathbf{X}^T(\mathbf{R}_0^T\mathbf{E} + \mathbf{E}^T\mathbf{R}_0)\mathbf{X} + 2(\mathbf{R}_0^T\mathbf{T}_0)^T(\mathbf{R}_0^T\mathbf{E})\mathbf{X} = 0. \tag{4.10}$$

It has been shown [6] that if and only if $\mathbf{R} = \mathbf{R}_0$ and $\mathbf{T} \times \mathbf{T}_0 = 0$, (4.10) will become trivial. Therefore (4.10) implies that unless the points used for correspondences lie on a quadric surface, called a Maybank quadric, no spurious solution with nonzero translation can produce the projections of the image correspondence points. We then have the following theorem, whose proof is presented in the Appendix.

THEOREM 4.2. *If the true motion is* $\mathbf{R}_0$ *and* $\mathbf{T}_0$, *then a sufficient condition for excluding any spurious solution* $\mathbf{R}$ *and* $\mathbf{T}$ *such that* $\mathbf{R} \neq \mathbf{R}_0$ *and* $\mathbf{T} \neq \mathbf{T}_0$ *is having six correspondences of points that do not lie on a Maybank quadric.*

The above theorem gives a condition which is sufficient but not necessary. A necessary and sufficient condition is generally expressed in nonlinear equations such as those in (2.13), (2.16), or (2.18), which are no easier than the motion epipolar line equation.

A Maybank curve is a degenerate form of Maybank quadric; that is, a Maybank curve is the interception of a Maybank quadric with a plane passing through the origin. Therefore, if the space points do not lie on a Maybank quadric, their projections in the image plane cannot lie on a Maybank curve. Summarizing the discussion in this section and the last section, we have the following theorem.

THEOREM 4.3. *Six correspondences of image points that do not correspond to space points lying on a Maybank quadric suffice to determine a motion uniquely.*

Recall that in the pure rotation case we only need five correspondences of image points that do not lie on a Maybank curve to determine the motion. But when there is a translation, we need one more correspondence and the requirement that the image points do not correspond to space points lying on a Maybank quadric. Whether the image points lie on a Maybank curve can be tested before solving for the motion, but whether the image points correspond to space points lying on a Maybank quadric cannot be tested before solving for the motion. Thus, when the translation is not zero, we so far have no means to ensure that the motion is uniquely determined by a given number of correspondences before we obtain one solution of it.

It is almost certain that five points define at least one Maybank quadric, as shown in the proof of Theorem 4.2 (see, also, the example in [22]). Therefore, if it is ever possible that five correspondences give a unique solution of motion, the probability for this to occur is zero. We will now discuss the following problem: if the Maybank quadric is uniquely defined, how many spurious solutions can the surface sustain? This is equivalent to Longuet-Higgins' problem of multiplicity of surface interpretations. We will show that at most one spurious solution can be sustained.

Assume $\mathbf{R}_0$, $\mathbf{T}_0$ represent the true motion and that $\mathbf{R}_1$, $\mathbf{T}_1$ and $\mathbf{R}_2$, $\mathbf{T}_2$ are two sets of spurious solutions such that $\mathbf{R}_i \neq \mathbf{R}_j$, $\mathbf{T}_i \times \mathbf{T}_j \neq 0$, for any $i \neq j$, $i, j \in \{0, 1, 2\}$. Other situations have been discussed in Lemma 4.1 and Lemma 4.2. Then for a given correspondence Θ' and Θ in the image plane, we should have

$$Z'\Theta' = Z\mathbf{R}_0\Theta + \mathbf{T}_0, \tag{4.11}$$

where Z' and Z are the true depths for Θ' and Θ, respectively. Similarly, for the two spurious solutions, there must exist positive numbers D', D, ρ', and ρ such that

$$D'\Theta' = D\mathbf{R}_1\Theta + \mathbf{T}_1, \tag{4.12}$$

$$\rho'\Theta' = \rho\mathbf{R}_2\Theta + \mathbf{T}_2. \tag{4.13}$$

From (4.12) and (4.13), we have the following *motion epipolar line equations*:

$$\Theta'^T(\mathbf{T}_1 \times \mathbf{R}_1)\Theta \triangleq \Theta'^T(\mathbf{E}_1)\Theta = 0 \tag{4.14}$$

and

$$\Theta'^T(\mathbf{T}_2 \times \mathbf{R}_2)\Theta \triangleq \Theta'^T(\mathbf{E}_2)\Theta = 0. \tag{4.15}$$

Now multiplying Eqs. (4.14) and (4.15) by the true depths Z' and substituting $Z'\Theta$ by (4.11), we obtain

$$Z\Theta^T(\mathbf{R}_0^T\mathbf{E}_1)\Theta + \mathbf{T}_0^T\mathbf{E}_1\Theta = 0, \tag{4.16}$$

$$Z\Theta^T(\mathbf{R}_0^T\mathbf{E}_2)\Theta + \mathbf{T}_0^T\mathbf{E}_2\Theta = 0. \tag{4.17}$$

Each of (4.16) and (4.17) defines a Maybank quadric. They must give the same depth for any correspondence except one, as the depths Z' and Z for all correspondences except one are uniquely determined by (4.11). The exception occurs when $\Theta' \times \mathbf{T}_0 = 0$. But in this case, the translation is uniquely determined, and Lemma 4.2 shows that five correspondences of points that do not lie on a Maybank curve uniquely determine the rotation. Therefore the motion in the exceptional case can be uniquely determined. In the discussion below we assume that such exceptions do not occur for all motion solutions and correspondence data. That is, $\mathbf{T}_i \times \Theta' \neq 0$ for any available correspondence Θ' and any $\mathbf{T}_i$, $i = 0, 1, 2$.

Since (4.16) and (4.17) must give the same depth for any correspondence, they must be identical. That means we must have

$$\mathbf{T}_0^T\mathbf{E}_1 = \mathbf{T}_0^T\mathbf{E}_2, \tag{4.18}$$

and

$$\mathbf{R}_0^T\mathbf{E}_1 + \mathbf{E}_1^T\mathbf{R}_0 = \mathbf{R}_0^T\mathbf{E}_2 + \mathbf{E}_2^T\mathbf{R}_0. \tag{4.19}$$

Rewriting (4.19) we obtain

$$\mathbf{R}_0^T(\mathbf{E}_1 - \mathbf{E}_2) = -(\mathbf{E}_1 - \mathbf{E}_2)^T\mathbf{R}_0. \tag{4.20}$$

The above equation shows that $\mathbf{R}_0^T(\mathbf{E}_1 - \mathbf{E}_2)$ must be skew-symmetric. Lemma 2.3 states that there must exist some vector $\mathbf{T}_3$ such that

$$\mathbf{E}_1 - \mathbf{E}_2 = \mathbf{T}_3 \times \mathbf{R}_0. \tag{4.21}$$

Then from (4.18) we know that $\mathbf{T}_0$ must be parallel to $\mathbf{T}_3$ since

$$\mathbf{T}_0^T(\mathbf{E}_1 - \mathbf{E}_2) = \mathbf{T}_0^T(\mathbf{T}_3 \times \mathbf{R}_0) = 0. \tag{4.22}$$

Therefore we can assume that $\mathbf{T}_3 = \alpha_0\mathbf{T}_0$. Then (4.21) gives

$$\mathbf{E}_1 - \mathbf{E}_2 = \alpha_0\mathbf{T}_0 \times \mathbf{R}_0 = \alpha_0\mathbf{E}_0. \tag{4.23}$$

For the same reason, for $\mathbf{R}_1$ and $\mathbf{T}_1$ to constitute a valid motion, we must have the following equations:

$$D\Theta^T(\mathbf{R}_1^T\mathbf{E}_0)\Theta + \mathbf{T}_1^T\mathbf{E}_0 = 0, \tag{4.24}$$

$$D\Theta^T(\mathbf{R}_1^T\mathbf{E}_2)\Theta + \mathbf{T}_1^T\mathbf{E}_2 = 0. \tag{4.25}$$

From (4.24) and (4.25) we have

$$\mathbf{E}_2 - \mathbf{E}_0 = \alpha_1\mathbf{E}_1 \tag{4.26}$$

for some constant α_1. Similarly,

$$\mathbf{E}_0 - \mathbf{E}_1 = \alpha_2\mathbf{E}_2. \tag{4.27}$$

For all three solutions to be acceptable simultaneously, (4.23), (4.26), and (4.27) must hold at the same time. If any constant α_i, $i = 0, 1, 2$, is zero, then one of the solutions must be identical to another, contradicting the assumption that the three solutions are distinct. So in the following we assume $\alpha_i \neq 0$, $i = 0, 1, 2$. Adding (4.23), (4.26), and (4.27) gives

$$0 = \alpha_0 \mathbf{E}_0 + \alpha_1 \mathbf{E}_1 + \alpha_2 \mathbf{E}_2. \quad (4.28)$$

Comparing (4.28) with (4.23) we have

$$(\alpha_1 + 1)\mathbf{E}_1 = (1 - \alpha_2)\mathbf{E}_2. \quad (4.29)$$

It is well known that an essential matrix can have at most two decompositions of the form $\mathbf{V} \times \mathbf{U}$ with $\mathbf{V}$ a nonzero vector and $\mathbf{U}$ a rotation matrix (see also the Appendix). And if $\mathbf{V}_1 \times \mathbf{U}_1$ and $\mathbf{V}_2 \times \mathbf{U}_2$ are two distinct decompositions of an essential matrix then we must have $\mathbf{V}_1 = -\mathbf{V}_2$. Since $\mathbf{E}_1$ and $\mathbf{E}_2$ are all essential matrices and $\mathbf{T}_1 \times \mathbf{T}_2 \neq 0$ by assumption, then we must have

$$\alpha_1 = -1, \quad \alpha_2 = 1. \quad (4.30)$$

But comparing (4.28) with (4.26) we also have

$$\alpha_0 = 1, \quad \alpha_2 = -1, \quad (4.31)$$

which contradicts (4.30). Therefore, there exist no three motion solutions which are mutually compatible. As the true motion solution must always hold, therefore there is at most one other solution which gives an alternate interpretation of the surface and the motion.

As it is possible that six points uniquely define a Maybank quadric or exclude the possibility that they lie on any Maybank quadric, to summarize the discussion above, we have the following theorem.

THEOREM 4.4. *When a translation is not zero, if six pairs of image points correspond to space points that uniquely define a Maybank quadric, then the six correspondences suffice to determine the motion parameters to within two sets; if the six pairs of points correspond to space points that do not lie on a Maybank quadric, then the six correspondences suffice to determine the motion uniquely.*

5. SUFFICIENT CONDITIONS FOR UNIQUE OR DOUBLE SOLUTIONS OF PLANE MOTION

When the points used for correspondences lie on a plane, the eight-point linear algorithm will fail. For a given plane, although it might not pass through the origin of the coordinate system and $-\mathbf{R}_0^T\mathbf{T}_0$, where $\mathbf{R}_0$ and $\mathbf{T}_0$ are the true motion parameters, we can always find another plane which passes through the above two points such that the two planes construct a quadric surface and hence violate the requirement of the applicability of the linear algorithm. Therefore we can only rely on plane motion algorithms or nonlinear algorithms to solve for plane motion. A question that immediately arises is whether correspondences of coplanar points admit motion solutions leading to nonplanar surface structures. We will now discuss this question.

It has been shown [8, 29, 18, 26] that when the points are coplanar in space, there generally exists a dual motion which always accompanies the true motion and cannot be removed with a rigidity constraint; unless the dual motion is identical with the true motion or yields negative depths for some visible points, the dual motion cannot be removed. Even worse, there exists an uncertain situation where an infinite number of motion solutions arise, although this situation occurs with zero probability [28, 26]. Longuet-Higgins has proven [28] that a planar surface admits spurious motion solutions leading to only planar surfaces. However, how many correspondences are needed and what conditions they should satisfy to exclude other solutions leading to non-planar surfaces are questions that still need to be answered. In the following we shall show that six or more correspondences satisfying a certain condition suffice to exclude all spurious solutions leading to non-planar surfaces and admit only the dual motion as a spurious solution, provided that the uncertain situation does not occur.

Assume that a plane in the space has an equation

$$\mathbf{N}_0^T\mathbf{X} = 1 \quad (5.1)$$

and is subject to a motion $\mathbf{R}_0$ and $\mathbf{T}_0$. Equation (5.1) is the non-degeneracy condition of the projection of a plane [26]. Only when the projection of a plane is not degenerate in both image planes is it possible to find point correspondences. Then it is well known that an image point correspondence pair Θ' and Θ are interrelated by

$$\gamma\Theta' = (\mathbf{R}_0 + \mathbf{T}_0\mathbf{N}_0^T)\Theta \triangleq \mathbf{K}\Theta, \quad (5.2)$$

where

$$\gamma = \sqrt{\Theta^T\mathbf{K}^T\mathbf{K}\Theta/\Theta'^T\Theta'} \quad (5.3)$$

and

$$\mathbf{K} = \mathbf{R}_0 + \mathbf{T}_0\mathbf{N}_0^T. \quad (5.4)$$

A matrix $\mathbf{K}$ of the type (5.4) is called a *plane motion matrix*, and $\mathbf{R}_0$, $\mathbf{T}_0$, and $\mathbf{N}_0$ are called the motion decomposition of $\mathbf{K}$. It has been shown (Hu [26], Tsai [8]) that unless $\mathbf{R}^T\mathbf{T}$ is parallel to $\mathbf{N}$, the plane motion matrix in (5.4) will still have another decomposition $\mathbf{R}_d$, $\mathbf{T}_d$, and $\mathbf{N}_d$, called the dual solution, with $\mathbf{N}_d \times \mathbf{N}_0 \neq 0$. Therefore, as we have shown in [26], the dual solution cannot be removed by the motion model and rigidity constraint, although it can sometimes be identified using a positive depth constraint [29]. There exists an uncertain situation, where $K^TK = I$ and $\det(K) = -1$. In this case, $\mathbf{K}$ has an infinite number of plane motion decompositions [26]. For this to occur, either the object is transparant and rotated by a half revolution or one of the images is taken through a mirror; we assume this situation does not occur in the discussion below. That is, we assume $\mathbf{K}$ has at most two plane motion decompositions.

First let us consider a spurious solution with a zero translation. Suppose there is a pure rotation $\mathbf{R}$ which produces the same image correspondences as $\mathbf{K}$ does. Then we should have

$$\beta\Theta' = \mathbf{R}\Theta, \tag{5.5}$$

where

$$\beta = \|\Theta\|/\|\Theta'\|; \tag{5.6}$$

(5.6) and (5.2) imply that

$$\left(\mathbf{K} - \frac{\gamma}{\beta}\mathbf{R}\right)\Theta = 0 \tag{5.7}$$

or

$$\frac{\gamma}{\beta}\Theta = \mathbf{R}^T\mathbf{K}\Theta. \tag{5.8}$$

The above equation indicates either

$$\mathbf{K} = \mathbf{R}, \quad \gamma \equiv \beta, \tag{5.9}$$

or Θ is an eigenvector of $\mathbf{R}^T\mathbf{K}$. Assume $\mathbf{R}^T\mathbf{K} \neq \mathbf{I}$. Then there are at most three vectors Θ_i, $i = 1, 2, 3$, of the form $(x, y, 1)^T$ such that

$$\mathbf{R}^T\mathbf{K}\Theta_i = \lambda_i\Theta_i, \quad i = 1, 2, 3, \tag{5.10}$$

where λ_i, $i = 1, 2, 3$, are the eigenvalues of $\mathbf{R}^T\mathbf{K}$. But (5.9) must hold for all correspondence data. Consequently, any four correspondences of points suffice to exclude the possibility that $\mathbf{R}^T\mathbf{K} \neq \mathbf{I}$. But when $\mathbf{R}^T\mathbf{K} = \mathbf{I}$, $\mathbf{R}$ is the same as $\mathbf{K}$. Therefore, *only the true solution is admitted*. As a result, any four correspondences of points suffice to exclude a pure rotation as a spurious solution.

Now let us consider a spurious motion solution $\mathbf{R}$ and $\mathbf{T}$ with $\mathbf{T} \neq 0$. Then for any image correspondence pair Θ' and Θ, there exist two positive numbers Z' and Z such that

$$Z'\Theta' = Z\mathbf{R}\Theta + \mathbf{T}. \tag{5.11}$$

As a consequence, the motion epipolar line equation

$$\Theta'^T(\mathbf{T} \times \mathbf{R})\Theta = 0 \tag{5.12}$$

must hold. Now substituting (5.2) into (5.12) yields

$$\Theta^T\mathbf{K}^T(\mathbf{T} \times \mathbf{R})\Theta = 0. \tag{5.13}$$

The above equation again describes a quadratic curve in the image plane, but a more general one than the one described by (3.6). Equation (5.13) states that except for those spurious solutions which make $\mathbf{K}^T(\mathbf{T} \times \mathbf{R})$ skew-symmetric, all other spurious solutions with a nonzero translation can be eliminated by six correspondences of image points that do not lie on a quadratic curve. It can be directly verified that if

$$\mathbf{K} = \mathbf{R}_0 + \mathbf{T}_0\mathbf{N}_0^T = \mathbf{R}_d + \mathbf{T}_d\mathbf{N}_d^T, \tag{5.14}$$

then both $\mathbf{K}^T(\mathbf{T}_0 \times \mathbf{R}_0)$ and $\mathbf{K}^T(\mathbf{T}_d \times \mathbf{R}_d)$ will be skew-symmetric. Therefore, the dual motion cannot be eliminated. In the following we shall show that if $\mathbf{K}^T(\mathbf{T} \times \mathbf{R})$ is skew-symmetric, then $\mathbf{T}$ and $\mathbf{R}$ must represent either the true motion or the dual motion.

(5.2) and (5.11) lead to

$$\left(\frac{Z'}{\gamma}\mathbf{K} - Z\mathbf{R}\right)\Theta = \mathbf{T}. \tag{5.15}$$

Using $\mathbf{T}$ to cross multiply both sides of (5.15) we obtain

$$\left(\frac{Z'}{\gamma}\mathbf{T} \times \mathbf{K} - Z\mathbf{T} \times \mathbf{R}\right)\Theta = 0. \tag{5.16}$$

Now because $\mathbf{K}^T(\mathbf{T} \times \mathbf{R})$ is skew-symmetric, Lemma 2.3 states that there exists a vector $\mathbf{T}_1$ such that

$$\mathbf{T} \times \mathbf{R} = \mathbf{T}_1 \times \mathbf{K}. \tag{5.17}$$

Substituting (5.17) and (5.2) into (5.16) gives

$$(\mathbf{T} - \alpha\mathbf{T}_1) \times \Theta' = 0, \tag{5.18}$$

where $\alpha = \gamma Z/Z' > 0$; (5.18) must hold for all Θ'. Thus, with three non-colinear correspondences we can conclude that

$$\mathbf{T} - \alpha\mathbf{T}_1 = 0 \tag{5.19}$$

for some constant α. But (5.19) and (5.17) imply that

$$\mathbf{T} \times \left(\mathbf{K} - \frac{1}{\alpha}\mathbf{R}\right) \triangleq \mathbf{T} \times \mathbf{G} = 0; \tag{5.20}$$

(5.20) indicates that the rank of $\mathbf{G}$ is at most one. Therefore, we can find two vectors $\mathbf{T}_3$ and $\mathbf{N}$ such that

$$\mathbf{G} = \mathbf{T}_3\mathbf{N}^T. \tag{5.21}$$

Substituting (5.21) into (5.20) leads to either $\mathbf{G} = 0$ or $\mathbf{T} \times \mathbf{T}_3 = 0$.

In the following we shall assume $\mathbf{G} \neq 0$, but $\mathbf{T} \times \mathbf{T}_3 = 0$. The case $\mathbf{G} = 0$ can be treated similarly. Without loss of generality we assume $\mathbf{T}_3 = \mathbf{T}$. From (5.20) and (5.21) we can write

$$\mathbf{K} = \frac{1}{\alpha}\mathbf{R} + \mathbf{T}_3\mathbf{N}^T = \frac{1}{\alpha}(\mathbf{R} + \alpha\mathbf{T}_3\mathbf{N}^T) \quad (5.22)$$

for some positive constant α. Now $\mathbf{K}$ and $\mathbf{R} + \alpha\mathbf{T}_3\mathbf{N}^T$ are both plane motion matrices. Recall [18, 26] that a matrix $\mathbf{P}$ can be a plane motion matrix if and only if the eigenvalues λ_i of $\mathbf{P}^T\mathbf{P}$, $i = 1, 2, 3$, satisfy

$$0 < \lambda_1 \leq \lambda_2 = 1 \leq \lambda_3. \quad (5.23)$$

In other words, $c\mathbf{P}$ and $\mathbf{P}$ cannot both be plane motion matrices unless $c^2 = 1$. Therefore for both $(1/\alpha)(\mathbf{R} + \alpha\mathbf{T}_3\mathbf{N}^T)$ and $(\mathbf{R} + \alpha\mathbf{T}_3\mathbf{N}^T)$ to be plane motion matrix, we must have $\alpha^2 = 1$. Since α must be positive we have $\alpha = 1$. Hence

$$\mathbf{K} = \mathbf{R} + \mathbf{T}_3\mathbf{N}^T = \mathbf{R} + \mathbf{T}\mathbf{N}^T, \quad (5.24)$$

which shows that $\mathbf{R}$ and $\mathbf{T}$ belong to one of the plane motion decompositions of $\mathbf{K}$. Thus, only the true and the dual motions are admitted. The following theorem summarizes the discussion in this section.

THEOREM 5.1. *If the points used for correspondences are coplanar and the uncertain situation defined by* $\mathbf{K}^T\mathbf{K} = \mathbf{I}$ *and* $\det(\mathbf{K}) = -1$ *does not occur, then six correspondences of image points that do not lie on a quadratic curve in the image plane suffice to exclude all spurious motion solutions other than the true solution and the dual solution.*

6. THE EFFECT OF NOISE ON UNIQUENESS OF SOLUTION

The discussion so far assumes that there is no noise in the correspondence data. When noise is present in the data, the uniqueness condition can be affected severely. In this section, we will only give a qualitative analysis of the effect of noise on the uniqueness condition of the solution by considering three qualitatively different situations.

When the problem is overdetermined, that is, when the given noise will not make the motion undetermined, then noise may prohibit finding a solution consistent with all correspondence data. Therefore an optimal solution must be sought to minimize, say, the squared sum of the differences between the correspondences estimated from the motion parameters and the observed correspondences. But for a given optimality criterion, there might be multiple locally and even globally optimal solutions for a given number of correspondences. Therefore, an overdetermined problem may not have a unique solution. Even if only one globally optimal solution exists, unless one searches all the possible solutions, one may not necessarily obtain the best solution for the given criterion.

If the problem is underdetermined, for example, when insufficient correspondence data are available, noise generally cannot change the uniqueness condition of solution. That is, noise cannot make the solution unique. However, exceptions can occur in some extreme cases, for instance, when noise makes a motion with nonzero translation look like a pure rotation. Since the uniqueness condition for determining a pure rotation is less stringent than that for determining a general motion, noise can also make an undetermined problem determined.

The most complicated situation occurs when the problem is critically determined, i.e., when no correspondence may be neglected to determine the motion uniquely. In this case, both situations mentioned above can occur so that noise may severely affect the uniqueness condition and result in wrong estimates. For example, noise can make a planar surface look nonplanar and vice versa, or it can make a pure rotation look like a motion with nonzero translation and vice versa. Even if the problem is still determined, the solution could hardly be robust, as any change in the correspondence data must cause a visible change in the solution.

Therefore, to make the solution unique and robust, more than the minimum required number of correspondences are desired. We will not discuss this problem further here. How to make algorithms efficient and robust will be the subject of a forthcoming paper.

7. A NONLINEAR ALGORITHM AND SIMULATIONS

We have developed a robust nonlinear algorithm and run a number of simulations to verify the theoretical results in this paper. The experimental results are all consistent with the theoretical results. We will first briefly describe the algorithm and then present two examples. The algorithm is divided into two steps: first estimate the rotation matrix $\mathbf{R} = (r_{ij})$ nonlinearly and then estimate the translation vector $\mathbf{T}$ linearly.

Given n (≥ 5) correspondences $\mathbf{\Theta}_i = (x_i, y_i, 1)^T$ and $\mathbf{\Theta}_i' = (x_i, y_i', 1)^T$, $i = 1, 2, \cdots, n$, from the motion epipolar line equation we have

$$\begin{bmatrix} (\mathbf{\Theta}_1' \times \mathbf{R}\mathbf{\Theta}_1)^T \\ (\mathbf{\Theta}_2' \times \mathbf{R}\mathbf{\Theta}_2)^T \\ \vdots \\ (\mathbf{\Theta}_n' \times \mathbf{R}\mathbf{\Theta}_n)^T \end{bmatrix} \mathbf{T} \triangleq \mathbf{U}_n\mathbf{T} = 0. \quad (7.1)$$

The algorithm will search **R** and **T** such that the following optimality criterion is minimized:

$$S_1 = \|\mathbf{U}_n\mathbf{T}\|^2. \qquad (7.2)$$

Since **T** can be determined only to within a scalar, we can restrict $\|\mathbf{T}\| = 1$ without loss of generality. Now, for a given estimate **R**, the optimal estimate of **T** corresponds to the eigenvector of $\mathbf{U}_n^T\mathbf{U}_n$ associated with $\mathbf{U}_n^T\mathbf{U}_n$'s least eigenvalue λ_m, which is just the minimum value of S_1. Therefore, minimizing S_1 can be reduced to minimizing λ_m, which is a function of only the rotation matrix. The algorithm is thus divided into two steps: first search for **R** to minimize the least eigenvalue of $\mathbf{U}_n^T\mathbf{U}_n$; then estimate **T** in a closed form by solving for the eigenvector of $\mathbf{U}_n^T\mathbf{U}_n$ associated with the least eigenvalue.

The first step of the algorithm is nonlinear, but the second step is linear and gives a closed form. Therefore most of the computation is spent on the first step. Fortunately, only the rotation matrix **R** and, hence, only three unknowns are involved in the first step; thus even an exhaustive search algorithm can be applied to estimate **R**. To do this, we represent the rotation matrix by

$$\mathbf{R} = \mathbf{A}_X(\omega_X)\mathbf{A}_Y(\omega_Y)\mathbf{A}_Z(\omega_Z), \qquad (7.3)$$

where

$$\mathbf{A}_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega_X & -\sin\omega_X \\ 0 & \sin\omega_X & \cos\omega_X \end{bmatrix},$$

$$\mathbf{A}_Y = \begin{bmatrix} \cos\omega_Y & 0 & \sin\omega_Y \\ 0 & 1 & 0 \\ -\sin\omega_Y & 0 & \cos\omega_Y \end{bmatrix},$$

$$\mathbf{A}_Z = \begin{bmatrix} \cos\omega_Z & -\sin\omega_Z & 0 \\ \sin\omega_Z & \cos\omega_Z & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (7.4)$$

The algorithm exhaustively searches all possible values of ω_X, ω_Y, and ω_Z to obtain a rough estimate of the angles to within 1° precision, minimizing λ_m of $\mathbf{U}_n^T\mathbf{U}_n$. Then we refine the estimates of ω_X, ω_Y, and ω_Z to within a precision of 0.05° by searching in the neighborhood of the rough estimates, still minimizing λ_m. After the rotation matrix **R** is solved, we then estimate **T** in a closed form by solving for the eigenvector of $\mathbf{U}_n^T\mathbf{U}_n$ associated with λ_m.

If $\Theta_i' \times \mathbf{R}\Theta_i$ is close to zero for all correspondences, then **T** must be zero. To determine if the translation is zero or not, another criterion

$$S_2 = \sum_{i=1}^{n} (u_i - x_i')^2 + (v_i - y_i')^2 \qquad (7.5)$$

is used, where

$$u_i = \frac{r_{11}x_i + r_{12}y_i + r_{13}}{r_{31}x_i + r_{32}y_i + r_{33}}, \quad v_i = \frac{r_{21}x_i + r_{22}y_i + r_{23}}{r_{31}x_i + r_{32}y_i + r_{33}}. \qquad (7.6)$$

If S_2 is small, then the translation is zero; otherwise, not.

This algorithm, although inefficient, is feasible and can be implemented in parallel. In practice, one does not need to search angles larger than 90°, as when the rotation is too large, it is impossible to find the correspondences. In principle the algorithm can minimize any optimality criterion and obtain the globally optimal estimate, because it searches all possible values. Therefore it gives better performance than linear algorithms. It also requires a less stringent condition for a unique solution of motion parameters.

In the experiments presented below we assume that focal length $f = 500$, and the coordinate origin O lies at the center of the 512 × 512 image plane, that is, $O = (256, 256)$. The algorithm records five sets of rotation parameters separated sparsely and yielding the minimum values of S_1. It is found that the true motion parameters always yield a much smaller value of S_1 than other motion parameters in the experiments when six or more correspondences are used. The following are just two typical examples.

The first example shows that even four correspondences uniquely determine a pure rotation. We simulated a pure rotation with

$$\omega_X = 1.0°, \quad \omega_Y = 2.0°, \quad \omega_Z = 0.4°.$$

The four original points are

$$(x_1, y_1) = (100, 400), \quad (x_2, y_2) = (400, 100),$$
$$(x_3, y_3) = (400, 400), \quad (x_4, y_4) = (100, 100),$$

and the exact correspondences are

$$(x_1', y_1') = (78.718, 410.060),$$
$$(x_2', y_2') = (381.546, 111.761),$$
$$(x_3', y_3') = (380.951, 408.932),$$
$$(x_4', y_4') = (82.508, 106.615).$$

When the motion is estimated with the exact correspondences, the estimates are exactly the same as the true motion parameters. When the motion is recovered from the correspondences with truncated errors, that is, when the correspondences used are

$$(x_1', y_1') = (78.0, 410.0), \quad (x_2', y_2') = (381.0, 111.0),$$
$$(x_3', y_3') = (380.0, 408.0), \quad (x_4', y_4') = (82.0, 106.0),$$

the estimated angles are

$$\omega_X = 0.9°, \quad \omega_Y = 2.05°, \quad \omega_Z = 0.5°,$$

and the estimated translation is zero.

The next example illustrates that five correspondences uniquely determine a motion with nonzero translation. The true motion parameters are

$$\omega_X = 10.03°, \quad \omega_Y = 0.0°, \quad \omega_Z = 2.06°.$$
$$\mathbf{T} = (0.1, 0.2, 0.01)^T, \quad \hat{\mathbf{T}} \triangleq \frac{\mathbf{T}}{\max(0.1, 0.2, 0.01)}$$
$$= (0.5, 1.0, 0.05)^T.$$

The correspondences used are

$$\begin{aligned}
(x_1, y_1) &= (100, 100), & (x_1', y_1') &= (92.672, 154.742),\\
(x_2, y_2) &= (100, 400), & (x_2', y_2') &= (55.016, 440.138),\\
(x_3, y_3) &= (400, 100), & (x_3', y_3') &= (347.749, 101.813),\\
(x_4, y_4) &= (400, 400), & (x_4', y_4') &= (364.979, 433.230),\\
(x_5, y_5) &= (200, 200), & (x_5', y_5') &= (177.862, 242.340).
\end{aligned}$$

The associated depths are

$$Z_1 = 3.0, \quad Z_2 = 2.0, \quad Z_3 = 1.0, \quad Z_4 = 1.5, \quad Z_5 = 2.3.$$

The estimated motion parameters are

$$\omega_X = 10.0°, \quad \omega_Y = 0.0°, \quad \omega_Z = 2.0°,$$
$$\hat{T} = (0.500, 1.000, 0.050)^T.$$

In the second example, when the coordinates are truncated into integers, then even six correspondences give a quite different result. This means that the motion solution is much more sensitive to noise when the translation is not zero than when the translation is zero.

From the above results we see that the nonlinear algorithm gives a unique solution with much less stringent conditions, compared with the linear methods. Actually, nonlinear algorithms are generally more robust than linear algorithms. Although the nonlinear algorithm cannot give a closed-form solution, it can give the globally optimal solution if only the given problem is determined. If a problem is linearly determined, it must be nonlinearly determined, but not vice versa, as the above examples and the theorems in this paper show. A more detailed discussion on nonlinear algorithms will be presented in a forthcoming paper.

8. SUMMARY

Our main results in this paper can be summarized as follows:

1. When the motion is a pure rotation, five correspondences of points that do not lie on a Maybank curve in the image plane suffice to determine the motion uniquely.

2. When the motion involves a translation, six correspondences of image points that do not correspond to space points lying on a Maybank quadric suffice to determine the motion uniquely.

3. When the points all lie on a plane and the uncertain situation characterized by $\mathbf{K}^T\mathbf{K} = \mathbf{I}$ and $\det(\mathbf{K}) = -1$ does not occur, six correspondences of points that do not lie on a quadratic curve in the image plane suffice to restrict the motion to the true and to the dual plane motion solutions.

4. Each Maybank quadric can sustain at most two physically acceptable solutions. Therefore, if a Maybank quadric surface is uniquely defined by six or more space points, then the motion and the surface can be determined to within two solutions.

5. A nonlinear solution of the motion parameters is feasible and will generally give better performance than linear solutions, as linear solutions only constitute a subspace of nonlinear solutions. Although inefficient, a nonlinear method requires a less stringent condition for a unique solution of motion parameters and gives more robust results.

The importance for investigating degenerate surface situations should not be underestimated. Although strict Maybank quadrics rarely occur in reality, a surface patch (e.g., a human face) viewed at a distance can often be approximated by a Maybank quadric, or even a plane. Therefore, degeneracy has to be considered in any practical algorithm that estimates motion and structure.

The results of this paper are not constructive proofs and, hence, cannot be used to construct algorithms for determining the motion and surface interpretation uniquely or to within two sets. Practical nonlinear algorithms for solving motions of arbitrary rigid surfaces including planes and Maybank quadrics have been developed which comply with the results in this paper. The discussion of such algorithms is the subject of a forthcoming paper.

APPENDIX

In this appendix we will first establish the following lemma[1] and then prove Lemma 2.2 and Theorem 4.2.

[1] The proof here is a simplified version of the original, inspired by an anonymous reviewer.

LEMMA A.1. *A matrix* $\mathbf{E}$ *is an essential matrix if and only if there exists a nonzero vector* $\mathbf{T} = (t_1, t_2, t_3)^T$ *such that*

$$\mathbf{E}\mathbf{E}^T = \mathbf{G}_T\mathbf{G}_T^T = -\mathbf{G}_T^2 = -(\mathbf{T}\times)^2, \quad (A.1)$$

where

$$\mathbf{G}_T = \mathbf{T}\times = \begin{bmatrix} 0 & -t_3 & t_2 \\ t_3 & 0 & -t_1 \\ -t_2 & t_1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1^T \\ \mathbf{g}_2^T \\ \mathbf{g}_3^T \end{bmatrix}. \quad (A.2)$$

Proof. Let

$$\mathbf{E} = \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_2^T \\ \mathbf{e}_3^T \end{bmatrix}. \quad (A.3)$$

The necessary part is direct: if $\mathbf{E}$ is an essential matrix, there will be some nonzero vector $\mathbf{T}$ and a rotation matrix $\mathbf{R}$ such that $\mathbf{E} = \mathbf{T} \times \mathbf{R}$; and then $\mathbf{T}$ satisfies (A.1).

The sufficient part is simple, too, using the results of Huang and Faugeras [23]. Assume $\mathbf{T}$ is a nonzero vector satisfying (A.1). Then we can show that the eigenvalues of $\mathbf{E}\mathbf{E}^T$ are 0, $\|\mathbf{T}\|^2$, and $\|\mathbf{T}\|^2$. Assuming α is an eigenvalue of $\mathbf{E}\mathbf{E}^T$ and $\mathbf{V}$ is the associated eigenvector, then

$$\mathbf{E}\mathbf{E}^T\mathbf{V} = -\mathbf{T} \times (\mathbf{T} \times \mathbf{V}) = \alpha\mathbf{V} \quad (A.4)$$

or

$$-[(\mathbf{T} \cdot \mathbf{V})\mathbf{T} - \|\mathbf{T}\|^2\mathbf{V}] = \alpha\mathbf{V}, \quad (A.5)$$

from which we can conclude that either $\alpha = 0$ or $\alpha = \|\mathbf{T}\|^2$. Further, we know that the eigenvectors satisfy

$$\mathbf{V}/(\mathbf{T} \cdot \mathbf{V}) = \mathbf{T}/\|\mathbf{T}\|^2 \quad \text{or} \quad \mathbf{T} \cdot \mathbf{V} = 0. \quad (A.4)$$

$\|\mathbf{T}\|^2$ is a double eigenvalue since it corresponds to two independent eigenvectors. Therefore, according to Huang and Faugeras's theorem [23], $\mathbf{E}$ is an essential matrix. Q.E.D.

Now let us prove Lemma 2.2 in Section 2.

Proof of Lemma 2.2. Although we can prove the theorem constructively, to make use of the existing results and hence simplify the proof, we will follow the proof in Lemma A.1 and consider three different situations: 1. $t_it_jt_k \neq 0$; 2. $t_i = t_j = 0$ and $t_k \neq 0$; 3. $t_i = 0$ and $t_jt_k \neq 0$; where i, j, k are any three distinct indices among 1, 2, 3.

Equation (A.1) is equivalent to the equations

$$\|\mathbf{e}_i\|^2 = t_j^2 + t_k^2, \quad \mathbf{e}_i \cdot \mathbf{e}_j = -t_it_j, \quad i \neq j,\ i, j \in \{1, 2, 3\}. \quad (A.5)$$

We now consider three different situations in which (A.5) can be satisfied:

1. It has been shown [23] that when $\mathbf{e}_i \cdot \mathbf{e}_j \neq 0$, for any i, j, then the following equation is necessary for $\mathbf{E}$ to be essential:

$$\frac{\mathbf{e}_1}{\mathbf{e}_2 \cdot \mathbf{e}_3} + \frac{\mathbf{e}_2}{\mathbf{e}_3 \cdot \mathbf{e}_1} + \frac{\mathbf{e}_3}{\mathbf{e}_1 \cdot \mathbf{e}_2} = 0. \quad (A.6)$$

It now remains to show that when the above equation is satisfied, $\mathbf{E}$ can find an decomposition of $\mathbf{T} \times \mathbf{R}$. It suffices to show that there exist three nonzero numbers, t_i, $i = 1, 2, 3$, such that (A.5) is satisfied. To show this, we let t_i, $i = 1, 2, 3$, be so chosen that

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = -t_1t_2, \quad \mathbf{e}_1 \cdot \mathbf{e}_3 = -t_1t_3, \quad \mathbf{e}_2 \cdot \mathbf{e}_3 = -t_2t_3. \quad (A.7)$$

Assume that (A.6) is satisfied. Now using $\mathbf{e}_i$, $i = 1, 2, 3$, to have inner product with both sides of (A.6), we then have

$$\|\mathbf{e}_1\|^2 = \mathbf{e}_1 \cdot \mathbf{e}_1 = t_2^2 + t_3^2, \quad \|\mathbf{e}_2\|^2 = \mathbf{e}_2 \cdot \mathbf{e}_2 = t_1^2 + t_3^2, \quad \|\mathbf{e}_3\|^2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = t_1^2 + t_2^2. \quad (A.8)$$

Therefore, (A.5) is satisfied.

2. For the same reason as above, when

$$\|\mathbf{e}_i\| = \|\mathbf{e}_j\| > 0, \quad \|\mathbf{e}_k\| = 0, \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0, \quad i, j, k \in \{1, 2, 3\}, \quad (A.9)$$

then if and only if

$$t_i = t_j = 0,\ t_k = \pm\|\mathbf{e}_i\| = \|\mathbf{e}_j\|, \quad (A.10)$$

is Eq. (A.5) satisfied.

3. It is also direct to verify that in Eq. (A.5), to have a solution of the form (2.19) it is necessary and sufficient that (2.18) is satisfied. In this case, $\mathbf{T} = (t_1, t_2, t_3)^T$ is given by

$$t_i = 0, \quad t_j^2 = \|\mathbf{e}_k\|^2, \quad t_k^2 = \|\mathbf{e}_j\|^2, \quad t_jt_k = -\mathbf{e}_j \cdot \mathbf{e}_k. \quad (A.11)$$

Since the above three situations are the only possible cases in which (A.5) is satisfied, we have proven the lemma. Q.E.D.

We now prove Theorem 4.2.

Proof of Theorem 4.2. We first examine the surface shapes that can be represented by (4.10). First, the quadric must pass through the origin and $-\mathbf{R}_0^T\mathbf{T}_0$. Lemma 3.1 indicates that $\mathbf{R}_0^T\mathbf{E}$ is an essential matrix. From results in [12], we know that (4.10) can only represent a subset of quadrics (such as hyperbolic paraboloids, circular cylinder, and intersecting planes). There exist other quadrics, such as an elliptic sphere, a one or two sheet hyperboloid, and a cone, which cannot be represented by (4.10).

Given a quadric surface

$$\mathbf{X}^T(\mathbf{A} + \mathbf{A}^T)\mathbf{X} + 2\mathbf{B}^T\mathbf{X} = 0, \quad \text{(A.12)}$$

to express it in the form of (4.10), $\mathbf{B}$ must be dependent on $\mathbf{A}$ and $\mathbf{A}$ must be an essential matrix. That is, we must have some $\mathbf{T}$ and $\mathbf{R}$ such that

$$\mathbf{A} = \mathbf{T} \times \mathbf{R}, \quad \mathbf{B}^T = (\mathbf{R}_0^T\mathbf{T}_0)^T\mathbf{A}. \quad \text{(A.13)}$$

Therefore, only the elements of $\mathbf{A}$ can be free variables. But $\mathbf{A}$ can have only five free elements because the largest element of $\mathbf{A}$ can be normalized to unity and $\mathbf{A}$ must satisfy (2.13), (2.16), or (2.18) to be an essential matrix. It is then possible that five points define a finite number of Maybank quadrics of the type (4.10). It follows six points may exclude the possibility that they lie on any Maybank quadric, e.g., when the six points together with the coordinate origin and $-\mathbf{R}_0^T\mathbf{T}_0$ define a cone. The existence of such possibility can be shown through numerical examples (see the results in Section 7 and Example 4 in [31]). We now show that it is actually possible to determine any motion $\mathbf{R}_0$ and $\mathbf{T}_0$ with six pairs of correspondences.

In Theorem 5.1 we have shown that in the plane motion case six correspondences of image points that do not lie on a quadratic curve in the image plane admit only the true motion solution and the dual solution. It has been known [8, 26] that the dual solution will be identical to the true solution if $\mathbf{R}_0^T\mathbf{T}_0$ is parallel to $\mathbf{N}$, where $\mathbf{N}$ is the plane normal. Therefore, a given motion $\mathbf{R}_0$ and $\mathbf{T}_0$ can be uniquely determined by any six pairs of points that correspond to space points lying on a plane orthogonal to $\mathbf{R}_0^T\mathbf{T}_0$ and do not lie on a quadratic curve in the image plane. This completes the proof. Q.E.D.

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REFERENCES

1. G. R. Fowles, *Analytical Mechanics*, Holt, Rinehart, & Winston, New York, 1986.
2. R. M. Winger, *An Introduction to Projective Geometry*, Dover, New York, 1962.
3. D. H. Ballard and O. A. Kimball, Rigid body motion from depth and optical flow, *Comput. Vision Graphics Image Process.* **22,** 1983, 95–115.
4. R. M. Haralick and X. Zhuang, A note on "rigid body motion from depth and optical flow," *Comput. Vision Graphics Image Process.* **34,** 1986, 373–387.
5. X. Zhuang and R. M. Haralick, Rigid body motion and the optical flow image, in *The First Conf. on Artificial Intelligence Appl., Denver, Dec. 6–7, 1986.*
6. X. Zhuang, T. S. Huang, and R. M. Haralick, Two-view motion analysis: A unified algorithm, *J. Opt. Soc. Amer. A* **3,** No. 9, 1986, 1492–1500.
7. R. Y. Tsai and T. S. Huang, Uniqueness and estimation of three-dimensional motion parameters of rigid objects with curved surfaces, *IEEE Trans. Pattern Anal. Mach. Intell.* **6,** No. 1, 1984, 13–26.
8. R. Y. Tsai and T. S. Huang, Estimating three-dimensional motion parameters of a rigid planar patch, II. Singular value decomposition, *IEEE Trans. Accoust. Speech Signal Processing.* **ASSP-30,** No. 4, 1982.
9. R. Y. Tsai and T. S. Huang, Estimating three-dimensional motion parameters of a rigid planar patch. III. Finite point correspondences and the three-view problem, *IEEE Trans. Accoust. Speech Signal Process.* **ASSP-32,** No. 2, 1984.
10. J.-Q. Fang, and T. S. Huang, Solving three-dimensional small-rotation motion equations: Uniqueness, algorithms, and numerical results, *Comput. Vision Graphics Image Process.* **26,** 1984, 183–206.
11. T. S. Huang, Determining three-dimensional motion and structure from two perspective views, *Handbook of Pattern Recognition and Image Processing* (Young and Fu, Eds.), Chap. 14.
12. S. Negahdaripour, Multiple interpretations of the shape and motion of objects from two perspective images, *IEEE Trans. Pattern Anal. Mach. Intell.* **12,** No. 11, 1990, 1025–1039.
13. C. Longuet-Higgins, A computer algorithm for reconstructuring a scene from two projections, *Nature* **293,** 1981, 133–135.
14. C. Longuet-Higgins, The reconstruction of a scene from two projections—Configurations that defeat 8-point algorithm, in *The First Conf. on A.I. Appl., Denver, Dec. 5–7, 1984.*
15. C. H. Lee, Structure and motion from two perspective views via planar patch, in *ICCV-88, Dec., 1988, Florida,* pp. 158–164.
16. J. W. Roach and J. K. Aggarwal, Determining the movement of objects from a sequence of images, *IEEE Trans. Pattern Anal. Mach. Intell.* **6,** 1980, 554–562.
17. A. Mitchie, S. Seida, and J. K. Aggarwal, Using constancy of distance to estimate position and displacement in space, *IEEE Trans. Pattern Anal. Mach. Intell.* **10,** No. 4, 1988, 594–598.
18. J. Weng, N. Ahuja, and T. S. Huang, Motion and structure from point correspondences: A robust algorithm for planar case with error estimation, in *Proceedings, Inter. Conf. Pattern Recognition, Rome, Italy, 1988.*
19. J. Weng, N. Ahuja, and T. S. Huang, Closed-form solution + maximum likelihood: A robust approach to motion and structure estimation, in *IEEE Conf. Computer Vision and Pattern Recognition, Ann Arbor, June 1988,* pp. 381–386.
20. C. Jerian and R. Jain, Determining motion parameters for scenes with transplantation and rotation, *IEEE Trans. Pattern Anal. Mach. Intell.* **6,** No. 4, 1984, 523–530.
21. S. D. Blostein and T. S. Huang, Estimating motion from range data, in *Proceedings 1st Conf. on AI Applications, Dec. 1984, Denver, CO.*
22. O. D. Faugeras and S. Maybank, Motion from point matches: Multiplicity of solutions, *IEEE Conf. Computer Vision and Pattern Recognition, June 1989,* pp. 248–255.
23. T. S. Huang and O. D. Faugeras, Some properties of the *E* matrix in two-view motion estimation, *Pattern Recognit. Mach. Intell.* **11,** No. 12, 1989, 1310–1312.
24. X. Hu and N. Ahuja, *Motion Analysis I: Basic Theorems, Constraints, Equations, Principles and Algorithms,* Technical Note 89-1, Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, Jan. 1989.

25. X. Hu and N. Ahuja, Motion estimation under orthographic projection," *IEEE Transactions on Robotics and Automation*, Vol. 7, No. 6, pp. 848–853, Dec. 1991.

26. X. Hu and N. Ahuja, A robust algorithm for plane motion solution, in *Proceedings Internat. Conf. on Automation, Robotics and Computer Vision, 1990*.

27. S. J. Maybank, The angular velocity associated with the optical flow field arising from motion through a rigid environment, *Proc. R. Soc. London A* **401,** 1985, 317–326.

28. H. C. Longuet-Higgins, Multiple interpretations of a pair images of a surface, *Proc. R. Soc. London A* **418,** 1988, 1–15.

29. H. C. Longuet-Higgins, The visual ambiguity of a moving plane, *Proc. R. Soc. London B* **223,** 1985, 165–175.

30. *College Mathematics Handbook*, Shang Dong Pub. Bur., Ji Nan, China, 1985.

31. A. N. Netravali, T. S. Huang, A. S. Krishnakumar, and R. J. Holt, Algebraic methods in 3-D motion estimation from two-view point correspondences, *Internat. J. Imaging Systems Technol.* **1,** 1989, 78–99.