Image Models

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Models of spatial variation used in the computer processing of pictorial information are surveyed. Models of images depicting homogeneous textures are reviewed under the categories of pixel-based and region-based models. Pixel-based models are further divided into one-dimensional time series models, random field models, and syntactic models The random field models incorporate either global or local properties of an image. To put the role of the two low-level models in perspective, several high-level models making use of semantic information are also described.

Key Words and Phrases: image models, texture models, high-level models, low-level models, pixel-based models, one-dimensional models, random field models, global models, local models, syntactic models, region-based models, time series models, Markov models

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INTRODUCTION

In recent years it has become increasingly clear that the design of efficient computer algorithms for image analysis, processing, and synthesis can only be done using the framework of an image model. The performance of an algorithm is critically tied to the types of images it addresses and to the form of the image data.

In its basic form, a digital image is an array of picture elements (pixels), although image characteristics are often exploited to obtain concise representations. As an example, consider the binary images. The image regions may be large or small, elongated or compact; they may have smooth or jagged borders, mutually dependent or independent orientations and locations. These properties may be used to devise representations which are more economical than the original pixel array. A segmented image of Kansas farmland, for example, may be concisely described by specifying the shapes and locations of the individual farm fields (although this kind of representation would not be appropriate for an image of river delta).

Any representation achieves spatial compactness by making some properties explicit at the penalty of obscuring others. For example, if a representation includes the specification of the contours of image regions, perimeter computations are simple, but area computations are complex. The space and time requirements (efficiency) of an algorithm depend upon the nature of an image through the complexity of the chosen representation. For gray-level images, there are intensity variations within the image components. This only enhances the need for a concise image description, which must

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now specify shapes of the regions as well as their intensity distributions.

Only for images of a specified type can an evaluation and comparison of algorithms be carried out. Consider, for example, the dozens of edge detectors that have been developed, using a wide variety of mathematical and heuristic operators. Each can be shown to work to some extent on an assortment of natural images. Very little can be said about their relative performance until we can provide the spatial distribution of gray levels in the images to be operated upon. In image synthesis, specifying the characteristics desired in the final image assumes still greater importance. The role of an image model is to provide a precise description of the image characteristics necessary for an efficient design of image operations.

Images usually portray two-dimensional projections of three-dimensional scenes. A projection may encompass a variety of different characteristics of scene parts. For example, an image may record for each sample point in the scene scalar data, such as brightness, temperature, or elevation, or vector data such as color. The spatial variation of the image represents a small, selected part of the information about the original scene. This information may be interpreted at differing levels of abstraction according to the degree to which knowledge about the scene is used to supplement the raw data. Each of these interpretations is a valid image model having a certain level of semantic content. *High-level models* involve highly semantic descriptions, whereas *low-level models* provide concise abstractions of the spatial variation.

Much has been said about the relative merits of the two types of models and the superiority of one over the other. It is our opinion that there is no basic conflict between the two approaches. Because they differ fundamentally in what they choose to model, the differences in details are a natural outcome. Although semantics of visual information are the ultimate concern, even high-level semantic models must start with a raw interpretation of the data. This is because at any resolution, image structure can be identified only to a limited degree of detail, beyond which even complex regions are perceived to possess a characteristic uniformity, called texture. Thus low-level models must serve as the basis for all image modeling and are the primary concern of this paper. We concentrate on models for homogeneous images, images that do not exhibit any macrostructure. Such models have been used in applications such as image restoration and texture analysis and synthesis. Figure 1 shows several homogeneous natural textures, some of them in perspective.

Low-level image models have traditionally been categorized as either statistical or structural. Statistical models characterize an image in terms of its statistical properties, such as autocorrelation or cooccurrence. Structural models, on the other hand, describe an image by its structural primitives and their placement rules. This classification is not very useful, though, for if a structural model is not also statistical, the images it describes are too regular to be of interest. If a statistical model cannot reveal the image's basic structure, it is too weak to be of much help. A somewhat better division of image models follows.

(1) *Pixel-Based Models:* These models view individual pixels as the primitives of an image. Specification of the characteristics of the spatial distribution of pixel properties [HAWK70, MUER70] constitutes the image description.

(2) *Region-Based Models:* These models view an image as a set of subpatterns placed according to a given set of rules. Both the subpatterns and their arrangement may be defined statistically, and

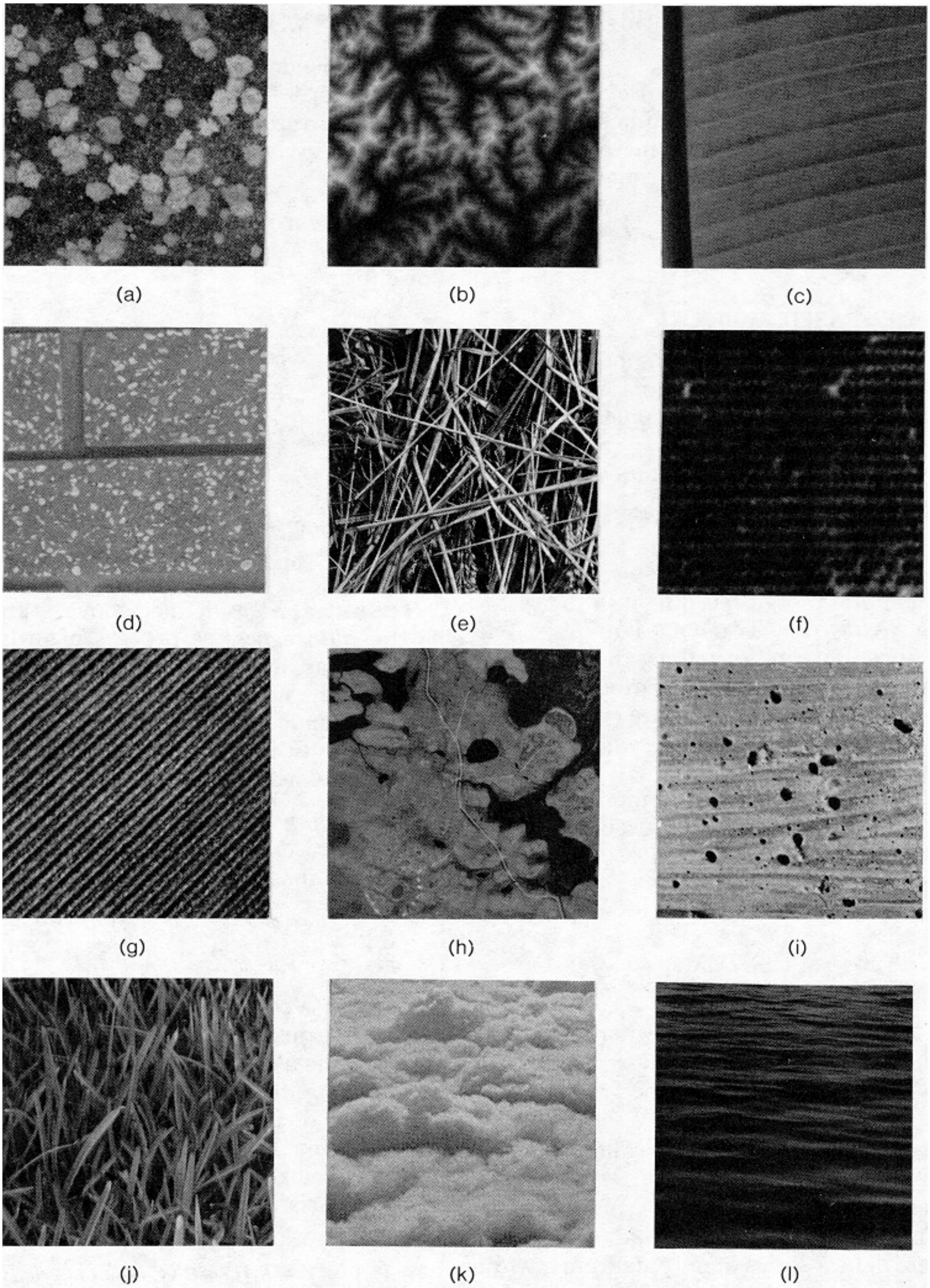


Figure 1. Some real textures (a) Volcanic rock, (b) terrain elevations, (c) leaf surface, (d) brick wall, (e) straw, (f) orchard, (g) plowed field, (h) mud tidal flats, (i) cement, (j) grass (in perspective), (k) cloud layer (in perspective), (l) water waves (in perspective)

these subpatterns themselves may be hierarchically composed of smaller patterns. Again, as with pixel-based models, the objective is to model a single texture; the concept of regions is used only to capture the microstructure.

In Sections 1 and 2 we discuss the low-level pixel-based and region-based models. In Section 3 we present some examples of high-level models, chosen primarily to illustrate the distinction between low-level and high-level models, with no attempt to survey the state of the art.

1. PIXEL-BASED MODELS

Pixel-based models can be subdivided into two classes: one-dimensional time series models and random field models.

1.1 One-Dimensional Time Series Models

Time series analysis [Box76] has been extensively used to model the statistical relationship between the gray level of a given pixel and of those preceding it in the raster scan [McCo74, Tou76a, Tou76b]. The gray-level fluctuations along the raster are treated as a stochastic process which evolves over time. The future course of the process is presumed to be predictable from information about its past.

Before summarizing the models, we review some commonly used notation [McCo74].

- Let

$$\cdots X_{t-1} X_t X_{t+1} \cdots$$

be a discrete time series where X_t is the random variable X at time i. We denote the series by $[X]$.

- Let μ be the mean of $[X]$, called the *level* of the process.
- Let $[\tilde{X}]$ denote the series of deviations about μ, that is, $\tilde{X}_t = X_t - \mu$.
- Let $[e]$ be a series of outputs from a white noise source with mean zero and variance σ^2.
- Let B be the "backward" shift operator for the deviation series such that $B\tilde{X}_t = \tilde{X}_{t-1}$; hence $B^m\tilde{X}_t = \tilde{X}_{t-m}$.
- Let ∇ be the backward difference operator for the deviation series such that

$$\nabla\tilde{X}_t = \tilde{X}_t - \tilde{X}_{t-1} = (1 - B)\tilde{X}_t.$$

Hence $\nabla^m\tilde{X}_t = (1 - B)^m\tilde{X}_t$.

The dependence of the current value $\tilde{X}_t$ on the past values of $\tilde{X}$ and e can be expressed in different ways, giving rise to several different models [McCo74].

(a) *Autoregressive Model* (*AR*). In this model the current $\tilde{X}$-value depends on the previous p $\tilde{X}$-values and on the current noise term

$$\tilde{X}_t = \phi_1\tilde{X}_{t-1} + \phi_2\tilde{X}_{t-2} + \cdots + \phi_p\tilde{X}_{t-p} + e_t. \quad (1)$$

If we let

$$\phi_p(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p,$$

then (1) becomes

$$[\phi_p(B)](\tilde{X}_t) = e_t.$$

The series $[\tilde{X}]$, as defined above, is known as the *autoregressive process* of order p, and $\phi_p(B)$ as the *autoregressive operator* of order p. The name "autoregressive" comes from the model's similarity to regression analysis and the fact that the variable $\tilde{X}$ is being regressed on previous values of itself.

(b) *Moving Average Model* (*MA*). In (a) above, $\tilde{X}_{t-1}$ can be eliminated from the expression for $\tilde{X}_t$ by substituting

$$\tilde{X}_{t-1} = \phi_1\tilde{X}_{t-2} + \phi_2\tilde{X}_{t-3} + \cdots + \phi_p\tilde{X}_{t-p-1} + e_{t-1}.$$

This process can be repeated to eventually yield an expression for $\tilde{X}_t$ as an infinite series in the e's.

A *moving average model* allows a finite number q of previous e-values in the expression for $\tilde{X}_t$. This explicitly treats the series as being observations on linearly filtered Gaussian noise.

Letting

$$\theta_q(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q,$$

we have

$$\tilde{X}_t = [\theta_q(B)](e_t)$$

as the *moving average process* of order q.

(c) *Mixed Model: Autoregressive/Moving Average (ARMA)*. To achieve greater flexibility in the fitting of actual time series, this model includes both the autoregressive and the moving average terms. Thus

$$\begin{aligned}\tilde{X}_t = \phi_1\tilde{X}_{t-1} + \phi_2\tilde{X}_{t-2} + \cdots \\ + \phi_p\tilde{X}_{t-p} + e_t - \theta_1 e_{t-1} \\ - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q},\end{aligned} \tag{2}$$

that is, $[\phi_p(B)](\tilde{X}_t) = [\theta_q(B)](e_t)$.

In all three models just mentioned, the process generating the series is assumed to be in equilibrium about a constant mean level. Models characterized by this equilibrium are called *stationary models*.

There is another class of models, *nonstationary models*, in which the level μ does not remain constant. The series involved may, nevertheless, exhibit homogeneous behavior after the differences due to level drift have been accounted for. It can be shown [Box76] that such behavior may be represented by a generalized autoregressive operator.

A time series may exhibit a repetitive pattern. For example, in a raster scanned image, the segments corresponding to rows will have similar characteristics. A model can be formulated that incorporates such "seasonal effects" [BACO65, McCO74].

All of the time series models discussed above are unilateral in that a pixel depends only upon the pixels preceding it in the raster scan. Partial two-dimensional dependence among pixels can be incorporated by letting a pixel depend upon a causal neighborhood [TOU76a, TOU76b, WHIT54], for example, upon that part of an $n \times n$ neighborhood centered at the pixel that precedes it in the raster scan [WHIT54]. This does not affect the applicability of the one-dimensional time series analysis. Whittle [WHIT54] points out that a one-dimensional approach has serious deficiencies. For example, even a finite bilateral autoregression may not always have a unilateral representation that is also a finite autoregression. Or the transformation to convert a bilateral dependence into a unilateral one may be prohibitively complex. Introduction of bilateral dependence gives rise to more complex parameter estimation problems [BART75, BROO64]. (Interestingly, a frequency domain treatment makes parameter estimation in bilateral representation much easier [CHEL79].)

1.2 Random Field Models

The theory of stochastic processes may be extended to define models of spatial variation in two dimensions. These models can be divided into two subclasses: global models and local models.

1 2.1 Global Models

Global models treat an entire image as the realization of a random field. The most common approach is to view an image as an ideal signal which has been corrupted by blurring and additive noise. The blur may represent a variety of spatial degradations [ANGE78, PRAT78a, ROSE76a]. For example, in aerial reconnaissance, astronomy, and remote sensing, the pictures obtained are degraded by atmospheric turbulence, aberrations of the optical system, and relative motion between object and camera. Electron micrographs are affected by the spherical aberration of the electron lens. Additive noise represents degradations that only affect the gray levels of individual points. A common example is thermal noise occurring in photodetectors. Film grain noise, which is multiplicative, can be converted to additive noise by subjecting the image to a logarithmic transformation.

Frieden [FRIE80] describes image restoration using global models. The problem involves solving the imaging equation

$$\mathbf{D} = S\mathbf{X} + \mathbf{n}$$

for the ideal image row $\mathbf{X}$, given the image data $\mathbf{D}$ and an estimate of the point spread matrix S, despite the presence of an additive noise component $\mathbf{n}$. Here $\mathbf{X}$, $\mathbf{D}$, and $\mathbf{n}$ are column vectors of n elements each, and S is an $n \times n$ matrix. Although inversion of the image formation equation is an unstable or ill-conditioned problem, a reasonable solution can be obtained with a priori knowledge of the nature of the true image. This information takes the form of constraints in the restoration procedure. Frieden treats the image as a spatial distribution of photons in cells centered at the grid nodes and then defines the image model as the process most likely to have generated the given distribution.

For restoration, images have often been modeled as two-dimensional, wide sense, stationary random fields having a given mean and autocorrelation function. The following general expression has been suggested for the autocorrelation function:

$$R(\tau_1, \tau_2) = \sigma^2 \rho^{[-\alpha_1|\tau_1| - \alpha_2|\tau_2|]}$$

which is stationary and separable. Specifically, the exponential autocorrelation function ($\rho = e$) has been found to be useful for a variety of pictorial data [FRAN66, HABI72, HUAN65, JAIN74, KRET52].

Another autocorrelation function often cited as being more realistic is

$$R(\tau_1, \tau_2) = \rho^{\sqrt{\tau_1^2 + \tau_2^2}}$$

which is isotropic, but not separable, as is the case with many natural images.

Hunt [HUNT76, HUNT77] points out that stationary Gaussian models are based upon an oversimplification. Consider the vector $\mathbf{x} = (x_1, \ldots, x_n)$ formed from sample values along a raster scan, with R being the covariance matrix of the gray levels in $\mathbf{x}$. According to the Gaussian assumption, the probability density function is given by

$$f(\mathbf{x}) = k \exp\{-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^t R^{-1}(\mathbf{x} - \boldsymbol{\mu})\}$$

where $\boldsymbol{\mu} = E[\mathbf{x}]$, and k is a normalizing constant. The stationarity assumption makes $\boldsymbol{\mu}$ a vector of identical components, meaning that each point in the image has identical ensemble statistics. Hunt [HUNT77] proposes a nonstationary Gaussian model which differs from the stationary model only in that the mean vector $\boldsymbol{\mu}$ has unequal components. He demonstrates the appropriateness of this model by subtracting the local ensemble average from each image point and showing that the resulting image fits a stationary Gaussian model.

In a later paper, Hunt [HUNT80] discusses three different kinds of nonstationarities:

- Case 1: Nonstationary mean, nonstationary autocorrelation.
- Case 2: Nonstationary mean, stationary autocorrelation.
- Case 3: Stationary mean, nonstationary autocorrelation.

Since a breakdown in stationarity implies a loss of the ergodicity assumption (loosely stated, a process is called "ergodic" if its statistics can be determined from any of the sample functions in the ensemble [ASSE78]), it is necessary to specify image statistics in terms of spatial averages, rather than ensemble averages. The nonstationarity in the image mean is described by the array of individual means taken over a specified neighborhood about each image point. For the autocorrelation function, the breakdown of stationarity is related to the way the correlation function changes over the image. Hunt uses three attributes of the correlation function to describe its spatial dependence: its energy, its width, and its shape. An image model consists of a specification of the parameters of a rotationally symmetric, negative exponential autocorrelation function, a set of means, and a spatial warp function to produce the autocorrelation nonstationarity.

Trussel and Kruger [TRUS78] claim that the Laplacian density function is better suited for modeling high-pass filtered imagery than the Gaussian function. Nevertheless, they contend that the basic assumptions which allow the Gaussian model to be used for image restoration purposes are still valid under a Laplacian model.

Nahi and Jahanshahi [NAHI77] suggest modeling the image by background and foreground statistical processes. The foreground consists of regions corresponding to the objects in the image. Each type of region (foreground or background) is defined by an associated statistical process. In estimating the boundaries of horizontally convex objects in noisy binary images, Nahi and Jahanshahi assume that the two processes are statistically independent stationary random processes with known (or estimated) first two moments. The borders of the regions covered by the different statistical processes are modeled only locally. The endpoints of the intercepts of the given object on successive rows are assumed to form a first-order Markov process. This model thus also involves local interactions.

Using the notation,

x_i = the gray level at image point i,
γ_i = a binary function carrying the boundary information at point i,
b_i = a sample gray level from the background process at point i,
o_i = a sample gray level from the foreground process at point i,
e_i = a sample gray level from the noise process at point i,

the model allows us to write

$$x_i = \gamma_i o_i + [1 - \gamma_i] b_i + e_i$$

where γ incorporates the Markov constraints on the object boundaries.

In a subsequent paper, Nahi and Lopez-Mora [NAHI78] use a more complex γ function. For each row, γ either indicates the absence of an object or provides a vector estimate of the object's width and geometric center in that row. Thus the two-element vector contains information about the object's size and skewness. The vectors corresponding to successive rows are assumed to define a first-order Markov process.

Cooper [COOP79] views the Nahi approach as too restrictive, in that it reduces the original two-dimensional problem to a one-dimensional problem. Cooper investigates the general class of images formed by a blob of constant gray level on a constant background, with additive white Gaussian noise over the entire image. He uses a Markov process to model the blob boundary and constructs a derivative field by taking the directional derivative at each pixel. He then estimates the blob boundary by maximizing the joint likelihood of a hypothetical blob boundary and of all the image data contained in the directional derivative field.

Recursive solutions based on differential (difference) equations are common in one-dimensional signal processing and have been generalized to two dimensions. Jain [JAIN77b] investigates the applicability of three kinds of random fields to the image-modeling problem, each characterized by a different class of partial differential equations (PDEs): hyperbolic, parabolic, and elliptic. A digital shape is defined by a finite difference approximation of a PDE. The class of hyperbolic PDEs is shown to provide more general causal models than autoregressive moving average models. For a given spectral density (or covariance) function, parabolic PDEs can provide causal, semicausal, or even noncausal representations. Elliptic PDEs provide noncausal models that represent two-dimensional discrete Markov fields and can be used to model both isotropic and anisotropic imagery. Jain argues that the PDE model is based on a well-established mathematical theory and, further, that there exists a considerable body of computer software for numerical solutions. The PDE model also obviates the need for spectral factorization, thus eliminating the restriction of a separable covariance function. System identification techniques may be used for choosing the PDE model for a given class of images. Chellappa [CHEL80] derives convexity properties of the autocorrelation function for the PDE models. The hyperbolic (elliptic) model gives an autocorrelation function that is convex (concave) along both axes. The autocorrelation function for the parabolic model is convex along one axis and concave along the other.

Matheron's [MATH71] regionalized random variable approach emphasizes pixel properties whose complex mutual correlation reflects the spatial structure. He assumes weak stationarity of the gray-level increments between pixels. The variogram

$$\gamma(d) = E[(\text{gray level at } i - \text{gray level } j)^2],$$

where $d = |i - j|$, is the basic analytic tool. Huijbregts [HUIJ75] gives numerous examples of regionalized variables: in geology, the ore grade and thickness, gravity, geochemical content; in forestry, the density of trees; in hydrology, the piezometric height; in meteorology, the quantity of dust and water vapor in the atmosphere. He discusses several properties of the variogram and relates them to the spatial structure of the regionalized variables. The variogram of the residuals with respect to the local mean is used for nonhomogeneous fields with locally varying mean.

Pratt and Faugeras [PRAT78b] and Gagalowicz [GAGA78] view texture as the output of a homogeneous spatial filter excited by white, not necessarily Gaussian, noise. A texture is characterized by its mean, the histogram of the input white noise, and the transfer function of the filter. For a given texture the model parameters are obtained as follows:

(1) The mean is readily estimated from the texture.
(2) The autocorrelation function is computed to determine the magnitude of the transfer function.
(3) Higher order moments are computed to determine the phase of the transfer function.

Inverse filtering yields the white noise image and hence its histogram and probability density. The inverse filtering or decorrelation may be done by simple operators. For example, for a first-order Markov field, decorrelation may be achieved by using a Laplacian operator [PRAT78b]. The whitened field estimate of the independent, identically distributed noise process will only identify the spatial operator in terms of the autocorrelation function, which is not unique. Thus the white noise probability density and spatial filter do not, in general, make up a complete set of descriptors [PRAT78c]. (To generate a texture, the procedure can be reversed by generating a white noise image having the computed statistics and then applying the inverse of the whitening filter.)

A random field may represent variation in gray level, color, elevation, or temperature, among other characteristics. Several researchers have proposed models specifically for height fields. One example is the Longuet-Higgins' model [LONG52, LONG57a, LONG57b] developed for the ocean surface (see also PIER52). Longuet-Higgins treats the ocean surface as a random field satisfying the following assumptions:

(1) the wave spectrum contains a single narrow band of frequencies;
(2) the wave energy results from a large number of different sources whose phases are random.

The basic equation governing the process is

$$X_{i,j} = \sum_k A_k \cos(u_k i + v_k j + \phi_k).$$

Longuet-Higgins [LONG57b] obtains the statistical distribution of wave heights. He also derives the relationships governing the root-mean-square wave height, the mean height of a given highest percentage of waves, and the most likely height of the largest wave in a given region.

Longuet-Higgins also obtains a set of statistical relations among parameters for (1) a random moving surface [LONG75a] and (2) a Gaussian isotropic surface [LONG57b]. Some of the results he obtains concern the following:

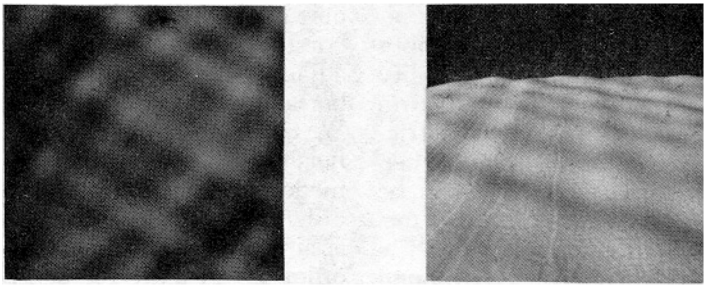
(1) the probability distribution of the surface elevation;
(2) the probability distribution of the magnitude and orientation of the gradient;
(3) the average number of zero crossings per unit distance along an arbitrarily placed line transect;
(4) the average contour length per unit area;
(5) the average density of maxima and minima;
(6) the probability distribution of the heights of maxima and minima.

All results are expressed in terms of the two-dimensional energy spectrum. He also studies and solves the converse problem: given certain statistical properties of the surface, determine a convergent sequence of approximations to the energy spectrum.

Longuet-Higgins' treatment of images of ocean waves makes use of features and analyses that may be used for image modeling in general. Panda [PAND78] adopts the Longuet-Higgins approach to analyze background regions selected from forward looking infrared (FLIR)[1] imagery. He derives expressions for the density of border points and average number of connected components along a row of a thresholded image and obtains generally good agreement between observed and predicted values. He uses the same approach [PAND79a] to predict the properties of images resulting from the application of edge detectors.

Schachter [SCHA80b] describes a variation of the Longuet-Higgins model. The

[1] A FLIR device uses an array of heat-sensing elements to record an image Infrared sensors are in wide use by the military, since infrared images can be made at night and do not require transmission of energy to be reflected



(a) (b)

Figure 2. (a) A texture generated using a long crested narrow-band noise model (b) The same texture displayed as a height field in perspective.

random field is described by the basic equation

$$X_{i,j} = \mu + \frac{1}{m} \sum_{k=1}^{m} A_k n(u_k i + v_k j + \phi_k), \qquad m \leq 3$$

where $n(\cdot)$ denotes a narrow-band noise waveform of a given center frequency and bandwidth, and m denotes the number of waveforms. A narrow-band noise waveform may be thought of as an envelope function modulating a carrier frequency. Thus this model describes textures of a two-level hierarchy; see Figure 2 for examples. The model has been implemented in hardware in a real-time image-generation system for flight simulation.

Several authors describe models for the earth's surface. Freiberger and Grenander [FREI76] reason that the earth's surface is too irregular to be represented by an analytic function which has only a small number of free parameters. Nevertheless, because landscapes possess strong continuity properties, they suggest using stochastic processes derived from physical principles. Mandelbrot [MAND77] uses a Poisson–Brown surface to give a first approximation to the earth's relief. The earth's surface is assumed to have been formed by the superimposition of very many, very small cliffs along straight faults. The positions of the faults and the heights of the cliffs are assumed random and independent. Mandelbrot suggests that the generated surface could be made to resemble some actual terrain more closely by introducing anisotropy into ridge directions. Mandelbrot's model is often used in computer graphics to generate artificial terrain scenes. Adler [ADLE76a–ADLE78c] presents a theoretical treatment of Brownian sheets and relates them to the rather esoteric mathematical concept of Hausdorff dimension. Mark [MARK77] discusses Brownian sheets and a number of other approaches to modeling the topological randomness of geomorphic surfaces.

1.2.2 Local Models

Global models characterize spatial variation in the whole image as a single random field. If neither knowledge nor hypotheses about the type of random field are available, a class of local models is used. Local models assume relationships among gray levels of pixels in small neighborhoods. Predetermined formalisms are used to describe such relationships. The modeling process consists of choosing a formalism and evaluating its parameters. Two basic categories of local models arise from the joint and conditional probability formulations of variation in a neighborhood proposed by Whittle [WHIT63] and Bartlett [BART55, BART67]. Whittle's definition requires that the joint probability distribution of the variables in a given neighborhood be of the product form

$$\prod_{i,j} Q_{i,j}(x_{i,j}; x_{i-1,j}, x_{i+1,j}, x_{i,j-1}, x_{i,j+1}, \ldots)$$

where $x_{i,j}$ is a realization of the random variable $X_{i,j}$ associated with pixel (i, j) and Q is a nonnegative function. Bartlett's definition requires that the conditional probability distribution of $X_{i,j}$ depend only upon the values at the neighbors of (i, j). Besag [BESA74b], however, describes a number of difficulties with this approach. There is no obvious way to obtain a joint probability structure for a given conditional probability model; the conditional probability structure is subject to some subtle, yet highly restrictive, consistency conditions, and when these restrictions are enforced, it can be shown [BROO64] that the conditional formulation is degenerate with respect to the joint formulation. Besag [BESA74b] found that the constraints on the conditional probability structure are so severe that they actually dictate particular models. (For example, for binary variables, the conditional probability formulation gives rise to the Ising model of statistical mechanics.)

The conditional probability approach, however, has served as the basis for a commonly used class of models, called Markov image models. Consider a finite image of n pixels with an associated collection of n random variables $\{X_i\}$. Suppose that for each pixel i, the conditional distribution of its associated random variable X_i, given all other pixel values, depends only upon those pixels within a finite neighborhood $N(i)$ of pixel i. Pixel j ($\neq i$) is said to be a *Markov neighbor* of pixel i if and only if the functional form $\Pr(x_i \mid x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is dependent upon the variable x_j. The complete set of neighborhoods $N_1, \ldots, N_n$ generate an associated class of valid probability distributions for $(X_1, \ldots, X_n)$. Any member of this class is formally called a *Markov*[2] (random) *field*. One result of the important Clifford–Hammersley theorem is that for any Markov field, the functional form

$$\Pr(X_i = x_i, X_j = x_j, \ldots, X_s = x_s \mid \text{all other pixel values})$$

depends only upon $x_i, x_j, \ldots, x_s$ and the values at neighboring sites of $i, j, \ldots, s$.

Different choices for the set of a pixel's neighbors give rise to different Markov models. A first-order (or nearest neighbor) Markov model lets the value at each pixel depend only upon the values at its 4-neighbors (two horizontal and two vertical neighbors). A second-order scheme uses an 8-adjacent neighborhood (the 3×3 neighborhood centered at the pixel), and so on. We let $N^m(i)$ denote the mth-order Markov neighborhood of pixel i. Second-order models are much more complex than first-order models, and unless the variables are assumed to be Gaussian, third- and higher order schemes are too cumbersome to exploit. First- and second-order schemes may be extended to other lattice structures [BESA74b], nonlattice structures [BESA75], higher dimensions, and the space–time domain [BESA74a], as well.

The specification of the probability distribution of a pixel's gray level, given the gray levels of its neighbors, defines a strict-sense Markov field representation [ROSE76a]. Besag describes two of the more interesting models of this kind [BESA74b], the autobinary and the autonormal schemes. The autobinary scheme has a conditional structure of the form

$$\Pr(x_i \mid x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = \frac{\exp[x_i(\alpha_i + \sum_{j\in N(i)} \beta_{i,j} x_j)]}{1 + \exp(\alpha_i + \sum_{j \in N(i)} \beta_{i,j} x_j)}$$

where x_i can take on only values of zero or one.

The autonormal scheme assumes that the joint distribution of the variables is multivariate normal. For this case, the conditional probability function is given by

$$Pr\ (x_i \mid x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (2\pi\sigma^2)^{-1/2} \exp[-\tfrac{1}{2}\sigma^{-2}\{(x_i - \mu_i)^t \times \sum_{j\in N(i)} \beta_{i,j}(x_j - \mu_j)\}^2].$$

And the joint density function is given by

$$\Pr(\mathbf{x}) = (2\pi\sigma^2)^{-n/2}\, |B|^{1/2} \times \exp[(\mathbf{x} - \boldsymbol{\mu})^t B(\mathbf{x} - \boldsymbol{\mu})/2\sigma^2],$$

where $\mathbf{x} = (x_i, \ldots, x_n)$, $\boldsymbol{\mu} = E[\mathrm{X}]$, and B is a matrix defined as follows:

$$b_{i,j} = b_{j,i} = \begin{cases} 1, & \text{if } \ i = j \\ -\beta_{i,j}, & \text{if } \ j \in N(i), \quad j \neq i \\ 0, & \text{otherwise.} \end{cases}$$

Also, $E[X_i \mid \text{all other pixel values}] = \mu_i + \sum_{j\in N(i)} \beta_{i,j}(x_j - \mu_j)$. For a first-order auto-

[2] It should be noted that the use of the term "Markov" in a two-dimensional context is somewhat controversial. Wong [WONG68], for example, offers a proof that there exists no continuous two-dimensional random field that is both homogeneous and Markov. Serra [SERR80] also discusses the misuse of the term "Markov" in digital image modeling

normal scheme, $X_{i,j}$ (given the values at all other pixels) is normally distributed with mean

$$\alpha + \beta_1(x_{i-1,j} + x_{i+1,j}) + \beta_2(x_{i,j-1} + x_{i,j+1})$$

and common variance σ^2.

For a discussion of other autoschemes, including the autobinomial, auto-Poisson, and autoexponential schemes, see Besag [BESA74b]. Cross and Jain [CROS81] report experiments on fitting the autobinomial model to textures from Brodatz [BROD66]. Maximum likelihood estimates of the model parameters are used to test the hypothesis that a given texture sample is described by an autobinomial model with the estimated parameters. The model is also used to generate texture samples.

A close relative of the automodel is the simultaneous autoregressive scheme, defined by

$$X_i = \mu_i + \sum_{j \in N(i)} \beta_{i,j}(X_j - \mu_j) + e_j$$

where the error (noise) terms are independent Gaussian variables with zero mean and common variance σ^2. The joint probability density is given by

$$\Pr(\mathbf{X}) = (2\pi\sigma^2)^{-n/2}|B| \times \exp[-2\sigma^{-2}(\mathbf{X} - \boldsymbol{\mu})^t B^t B(\mathbf{X} - \boldsymbol{\mu})]$$

where B is a nonsingular matrix defined as follows:

$$b_{i,j} = \begin{cases} 0, & \text{if } i = j, \\ -\beta_{i,j}, & \text{if } j \in N(i), \quad j \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

A first-order scheme of this type is given by

$$X_{i,j} = \beta_1 X_{i-1,j} + \beta_2 X_{i+1,j} + \beta_3 X_{i,j-1} + \beta_4 X_{i,j+1} + e_{i,j}.$$

For this simple model,

$$\begin{aligned} &E[X_{i,j} \mid \text{all other pixel values}] \\ &\quad = \alpha[(\beta_1 + \beta_2)(x_{i-1,j} + x_{i+1,j}) \\ &\qquad + (\beta_3 + \beta_4)(x_{i,j-1} + x_{i,j+1}) \\ &\qquad - (\beta_1\beta_4 + \beta_2\beta_3)(x_{i-1,j-1} + x_{i+1,j+1}) \\ &\qquad - (\beta_1\beta_3 + \beta_2\beta_4)(x_{i-1,j+1} + x_{i+1,j-1}) \\ &\qquad - \beta_3\beta_4(x_{i,j-2} + x_{i,j+2}) \\ &\qquad - \beta_1\beta_2(x_{i-2,j} + x_{i+2,j})] \end{aligned}$$

where $\alpha = 1/(1 + \beta_1^2 + \beta_2^2 + \beta_3^2 + \beta_4^2)$ [BESA74b] and $\mathrm{Var}(X_{i,j} \mid \text{all other pixel values}) = \alpha\sigma^2$.

Besag [BESA75] notes that the symmetry requirements of the simultaneous autoregressive model (i.e., $\beta_1 = \beta_2$; $\beta_3 = \beta_4$) are automatically fulfilled without the need to place prior restrictions on them. He also notes that $e_{i,j}$ and $x_{i',j'}$ are uncorrelated whenever $(i, j) \neq (i', j')$.

If the mean-squared error of fit in the autoregressive model is minimized, the resulting model is said to describe a wide-sense Markov field [ROSE76a]. The minimization of the error requires that the error terms at the various pixels be uncorrelated random variables. Woods [WOOD72] shows that the strict-sense Markov field defined earlier differs from a wide-sense field only in that the error variables in the former have a specific correlation structure, whereas the errors in the latter are uncorrelated. For the case of Gaussian random fields the two definitions are equivalent [ROSE76a]. Woods points out the restrictions on the strict-sense Markov field representation under which it yields a model for non-Markovian processes and also specifies the condition under which a general, noncausal, Markov dependence reduces to a causal one.

The method of least squares parameter estimation does not give consistent results for the simultaneous autoregressive scheme [BESA74b, BESA75, CLIF73, HEPP74, MEAD71, ORD75]. Furthermore, the error terms in the simultaneous autoregressive scheme must be treated as being correlated with the (noisy) image terms [BART75, BESA74b, BESA75, MORA73, PAND77]. Because of these two limitations, the use of the wide-sense Markov field model (common in image restoration) as a simultaneous autoregressive model is incorrect.

We have not covered techniques for estimating the unknown parameters of these models. Good discussions of this topic can be found in BALL77, BESA74b, BESA76, BESA77, BOX76, and ORD75. The relationship between causal and noncausal Markov neighborhoods has also been studied in two dimensions. Abend et al. [ABEN65] use Markov chain methods to show that in many cases a noncausal dependence is equivalent to a causal dependence. Woods' [WOOD72] treatment allows a larger number of equivalent pairs of causal and noncausal neighborhoods. And still other studies present additional examples and discussion of Markov models [DEGU76, HASS78, JAIN74, JAIN77a, PAND77, PICD77, STRA77, WELB77].

Kashyap [KASH80] uses circulant matrices to define autoregressive models for an infinite array (the array obtained by infinitely repeating a given finite image). This obviates the need for the initial conditions required by the nonperiodic autoregressive processes. Kashyap obtains the probability density of the data and discusses the maximum likelihood estimation of the model parameters. He then gives decision rules for the choice of neighbors of a pixel (see also CHEL81) and for testing the homogeneity of image data. For nonhomogeneous images, he uses multivariate autoregressive models which are constructed as follows. An image is divided into small blocks each of which is assumed to be homogeneous; a univariate autoregressive model is then obtained for each of the windows, and the parameter vectors of these models are then viewed as data and are modeled by a multivariate random field. Finally, Kashyap [KASH80] obtains an expression for the probability density of the field.

Links and Biemond [LINK79] (see also ANDR77 and PRAT75) have investigated the autocorrelation–separability of image models. They consider a general model of the form

$$X_i = \sum_{j \in N(i)} \beta_{i,j} X_j + \gamma e_j$$

or in vector–matrix notation

$$\underset{n\times 1}{\mathbf{X}} = \underset{n\times n}{B}\, \underset{n\times 1}{\mathbf{X}} + \gamma \underset{n\times 1}{\mathbf{e}}$$

where $n = M^2$. The model autocorrelation is clearly

$$E[XX^t] = \gamma^2 B^{-1} E[\mathbf{ee}^t](B^{-1})^t.$$

The $n \times n$ autocorrelation matrix $E[\mathbf{XX}^t]$ may be separable into a direct product of an $M \times M$ column autocorrelation matrix B_c and an $M \times M$ row autocorrelation matrix B_r; that is,

$$\begin{aligned} E[\mathbf{XX}^t] &= E[\mathbf{X}_c\mathbf{X}_c^t] \otimes E[\mathbf{X}_r\mathbf{X}_r^t] \\ &= \gamma^2(B_c^{-1} \otimes B_r^{-1})E[\mathbf{e}_c\mathbf{e}_c^t] \\ &\quad \otimes E[\mathbf{e}_r\mathbf{e}_r^t](B_c^{-1} \otimes (B_r^{-1})^t) \end{aligned}$$

where $\mathbf{X}_r = (X_{i,1}, \ldots, X_{i,M})$, $\mathbf{X}_c = (X_{1,j}, \ldots, X_{M,j})^t$.

Thus, for separability of the model autocorrelation, it is only necessary that the model operator be separable into a column operator B_c and a row operator B_r and that the error autocorrelation $E[\mathbf{ee}^t]$ be separable into a column autocorrelation $E[\mathbf{e}_c\mathbf{e}_c^t]$ and a row autocorrelation $E[\mathbf{e}_r\mathbf{e}_r^t]$.

If the elements of $\mathbf{e}$ are uncorrelated variables with zero mean and variance σ^2, the autocorrelation matrix $E[\mathbf{ee}^t]$ is separable: that is, $E[ee^t] = \sigma^2 I_c \otimes I_r$. The $\beta_{i,j}$ terms can be written in a matrix of the form

$$\begin{array}{ccccc} & \vdots & \vdots & \vdots & \\ \cdots & \beta_{-1,-1} & \beta_{-1,0} & \beta_{-1,1} & \cdots \\ \cdots & \beta_{0,-1} & -1 & \beta_{0,1} & \cdots \\ \cdots & \beta_{1,-1} & \beta_{1,0} & \beta_{1,1} & \cdots \\ & \vdots & \vdots & \vdots & \end{array}$$

If every two rows of the matrix are pairwise linearly independent, then the model autocorrelation is separable.

The local models discussed so far relate the gray level of a pixel to the gray levels of its neighbors, using a conditional probability formulation. The joint probability approach has also received considerable attention. Its primary difficulty, however, is the high dimensionality of the joint probability densities, even for small neighborhoods, making parameter estimation complex and cumbersome. Read and Jayaramamurthy [READ72] and McCormick and Jayaramamurthy [McCO75] make use of switching theory techniques to reduce this intractability. They develop minimal functions for modeling local gray-level patterns as follows. Suppose that each pixel is assigned one of N gray levels; then a neighborhood of M pixels may be represented by a point in an $(M \times N)$-dimensional space. Points corresponding to an aggregate of neighborhoods from a single pattern are likely to form clusters in this space. A generalization of standard switching theory, the set-covering methodology of Michalski and McCormick [MICH71] is used to describe the set of points in a cluster. These maximal descriptions allow coverage of empty spaces in and around clusters. The sample size need only be large enough to provide a reasonable representation. This approach has the advantage of a simple table look-up decision for classifying textures [HARA76a].

A number of investigators confine joint statistics to neighborhoods of size two. A texture is characterized in terms of its gray level cooccurrence tallies, which are the first estimate of the corresponding joint probabilities. Julesz [JULE62] uses this approach in a number of texture studies in visual perception. His goal is to determine the set of statistical properties that allow humans to discriminate between different textures. Rosenfeld and Troy [ROSE70b] and Haralick [HARA71, HARA73, HARA76a, HARA76c] suggest the use of two-dimensional spatial dependence of gray levels for fixed distances and angular separations. Haralick and numerous other investigators apply features derived from the cooccurrence matrix to various texture classification and discrimination problems. The performance of these features as texture measures is compared with several other techniques by Weszka, Dyer, and Rosenfeld [WESZ76] and Conners and Harlow [CONN80]. Zucker and Terzopoulos [ZUCK80] note that if a cooccurrence matrix is treated as a contingency table, then a standard chi-squared test can be used to see whether the rows and columns of the matrix are independent. (Presumably, any dependence found here is due to the structure in the image.) They use the maximum among chi-squared values corresponding to various choices of distance and orientation to identify structural characteristics of the image. Their particular approach has been used in geology for many years [HARB70, VIST65, WHIE75], though in geology the classes are separate entities (mineral types), not ordered gray levels. Davis et al. [DAVI79] suggest the use of a generalized cooccurrence matrix based upon local features rather than gray levels.

1.3 Syntactic Models

Rosenfeld [ROSE80] defines a *procedural model* as "any process that generates or recognizes images; the class that it defines consists of the images that it accepts." Grammatical (or syntactic) models fall into this category.

Conventionally, a *language* is defined as a set of strings over an alphabet, where the *alphabet* consists of the set of all symbols which can appear in the strings of the language, and a *string* is a finite ordered sequence of symbols. A *grammar* is a set of rules which define how the strings of the language are formed. Equivalently, a grammar can be used to recognize the language's strings by using the rules in reverse order. This concept can be generalized in a number of ways [FU73, LU78, ROSE71, ROSE79] to define grammars for classes of images.

An *array grammar* generates images by repeatedly replacing subarrays by other subarrays. A stochastic array grammar is one in which the replacement rules are probabilistic.

In one-dimensional grammars there is no problem with replacing one string by another. But for array grammars the shapes of subarrays must be compatible so that there are no "holes" in the resulting image. Jayaramamurthy [JAYA79] attempts to solve this problem with multilevel grammars. Here the subarrays at levels greater than zero are patterns of a specified shape, with the patterns themselves derived by a set of grammars at the next lower level. All the terminal subarrays at any level represent patterns of the same shape.

Syntactic methods have been used for locating highways and rivers in LANDSAT images and for texture modeling (see Fu [FU80] for additional references). Although the use of subarrays in the vocabulary of array grammars poses many difficulties, it gives these models a flavor of the second major class of models, and the region-based models.

2. REGION-BASED MODELS

Region-based models are defined using regions, instead of pixels, as primitives. A given model specifies the shapes of the regions and gives the rules for their placement in the plane, thereby allowing increased control over some pattern characteristics. Both the shapes and the placement rules may be specified statistically.

Over the years, region-based models have received far less attention than the pixel-based models in computer image processing. But recently these models have been investigated for texture analysis and synthesis. One class studied is that of mosaic models. These models view an image as a mosaic, constructed by tessellating the plane into cells and then coloring the cells by some given process. The resultant pattern consists of color patches, each patch formed by identically colored contiguous cells. Tessellations commonly used include the regular triangular, square, and hexagonal tessellations, and random tessellations such as the following:

(1) *Poisson line:* A Poisson process chooses pairs (ρ, θ), $0 \leq \theta \leq \pi$, $-\infty < \rho < \infty$. The lines $x \cos \theta + y \sin \theta = \rho$ define a tessellation of the plane.
(2) *Voronoi:* A Poisson process chooses points (nuclei) in the plane. Each nucleus defines a "Dirichlet cell" consisting of all the points in the plane nearer to it than to any other nucleus.
(3) *Delaunay:* All pairs of nuclei whose Dirichlet cells are adjacent are joined by straight line segments to define the tessellation.

The coloring process most commonly used assigns one of a predetermined set of colors to a cell, according to a given probability vector. The term "color" may stand for a fixed gray level or for gray levels from a given distribution, a vector of red, green, and blue components, among other possibilities.

Another class of region-based models consists of the coverage (or "bombing") models. These models view an image as a random arrangement of a given set of colored shapes over a uniform background. Once again, the choice of shapes and placement rules specifies a particular model. Circles have often been used for the figures, placed at locations chosen by a Poisson process. Figure 3 shows some examples of random mosaics and coverage patterns.

Region-based models of the above types, mosaic and coverage, have been popular in many disciplines, including geology, forestry, biology, ecology, astronomy, crystallography, and statistics. Several properties of mosaics and coverage patterns have been obtained by researchers in these fields. Many of the point correlation properties are discussed in Matern's [MATE60] excellent thesis on spatial variation. Switzer [SWIT65, SWIT67, SWIT69] and Pielou [PIEL77] extend Matern's results. They [SWIT65, PIEL77] show that a Poisson line mosaic formed from independently colored cells has the interesting property that any sequence of colors along a transect forms a Markov chain. The Markov properties of the Poisson line model have also been investigated by Scheaffer [SCHE75a, SCHE75b]. Switzer [SWIT65] derives an expression for the autocorrelation function of the Poisson line mosaic, and Matern [MATE69, MATE72] discusses some geometrical properties of the cells in random tessellations. Extensive work on estimating the geometrical characteristics of cells in Poisson line tessellations has been done by Miles [MILE64a, MILE64b], Crain and Miles [CRAI76], and Richards [RICH64]. They estimate properties such as the expected area of a cell and the expected number of sides of a cell, the expected perimeter of a cell, and the expected number of cells meeting at a vertex. Modestino et al. [MODE79a, MODE79b, MODE80] discuss joint pixel properties for mosaics with correlated cell colors. Zucker [ZUCK76] views texture as a distortion of an ideal mosaic. The distortion is specified in terms of geometric transforms applied to the mosaic.

Ahuja [AHUJ79, AHUJ81a, AHUJ81c] derives extensive results on the properties of connected color components in regular mosaics (triangular, square, and hexagonal) and in random mosaics (Poisson line, Voronoi, and Delaunay). Among the properties analyzed are the expected component area, the component perimeter, the component width, the component density, and the point correlation properties. The Voronoi mosaic is of special interest because its cells mimic those formed by natural growth processes. Matern [MATE60] gives

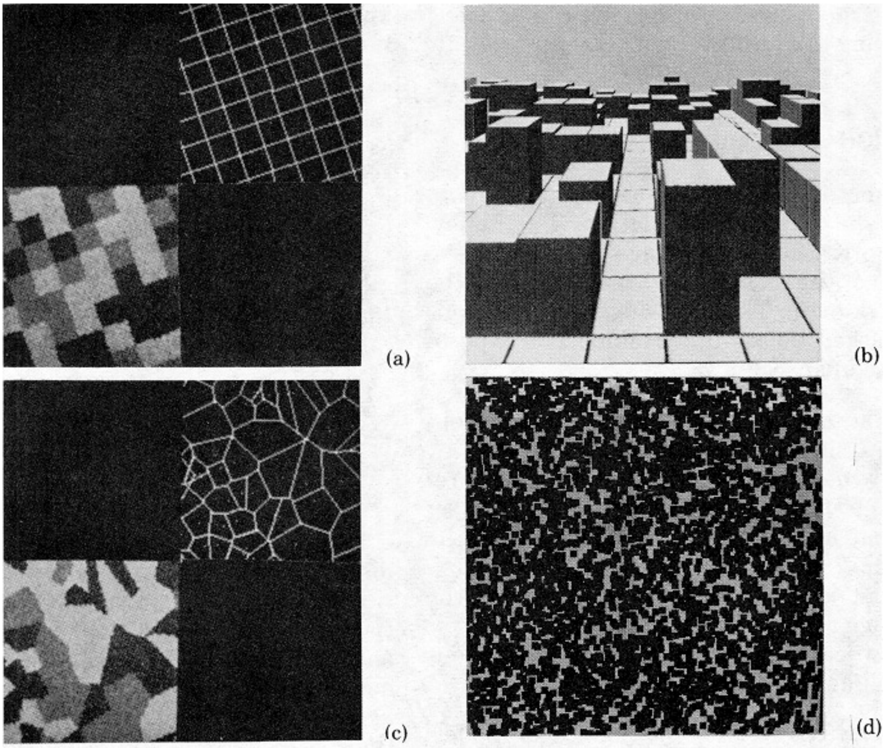


Figure 3. Examples of random mosaics and coverage patterns. (a) Square tessellation and mosaic. (b) Square mosaic displayed as a height field in perspective. (c) Voronoi tessellation and mosaic. (d) Square coverage pattern.

its autocorrelation function as a double integral equation having no elementary solution. Ahuja [AHUJ79] and Moore [MOOR78] provide empirical estimates. The geometrical characteristics of the Voronoi tessellation are covered by Miles [MILE70], Gilbert [GILB62], and Lee [LEE76]. Santalo [SANT76] is a good reference for the properties of various tessellations.

Solomon [SOLO53] first popularized the circular coverage model. Analogous to the analysis of mosaics, Ahuja [AHUJ79, AHUJ81b, AHUJ81c] obtains properties of the color components and joint characteristics of pairs of points for coverage patterns. Moore [MOOR73, MOOR74] has investigated anisotropic random mosaics. The application of region-based models to modeling images has received very little attention. Conners and Harlow [CONN80] start with a real-world texture and attempt to find a set of unit patterns and placement rules which could generate the texture. Ahuja et al. [AHUJ80, AHUJ81e] and Schachter et al. [SCHA78] attempt fitting mosaic models to a variety of textures. Both of them choose the model and its parameters so as to provide a match between the observed and predicted expected values of component properties. Modestino et al. [MODE79b] report experiments with texture discrimination using mosaic models. A detailed treatment of many aspects of mosaic and coverage models appears in AHUJ82.

Region-based models are further discussed by Rosenfeld and Lipkin [ROSE70a], Serra et al. [SERR73, SERR80], Miles [MILE80], Matheron [MATH67, MATH71], Ripley [RIPL76], Grunbaum and Shepard [GRUN82], Haralick [HARA78], Schachter and Ahuja [SCHA79], Ahuja and Rosenfeld [AHUJ81d], and Davis and Mitiche [DAVI80]. Syntactic models mentioned herein have also been used as region-based models with regions instead of pixels defining the terminal symbols.

3. HIGH-LEVEL MODELS

Higher level models treat an image as more than just a spatial variation. They allow interpretation of an image in terms of the goals of the analysis and the context of the data domain. The following basic questions are asked before undertaking any type of image analysis.

- What are the image primitives—pixels, lines, surfaces, regions, volumes?
- How does the available numeric data relate to these primitives?
- What information is available about the image?
- What is the best representation for the image and the knowledge base?
- How can the known information about the image be used?

We briefly review some high-level models resulting from work in a variety of fields: visual information processing, artificial intelligence, manufacturing, and military reconnaissance. The discussion here is only intended to be illustrative of the high-level approach, not a commentary on the state of the art. It is included to indicate the broad scope of the image-modeling problem and to help view the purpose and role of the low-level models from a more global perspective.

Marr and Poggio [MARR76] make the following observations about the physical world.

- A given point on a physical surface has a unique position in space at any one point in time.
- Matter is cohesive and is separated into objects whose surfaces are generally smooth in proportion to their distance from the viewer.

Barrow and Tenenbaum [BARR78] state that "the only hope of decoding the confounding information is, apparently, to make assumptions about the real world and exploit the constraints they imply." They give the following examples of real-world knowledge about scenes and visual perception.

- In the three-dimensional world, surfaces are often continuous and nearly uniformly reflectant.
- Step changes in intensity occur at surface or shadow boundaries.
- Perspective, texture gradient, and occlusion all provide clues to the understanding of a scene.
- In the man-made environment, straight edges frequently correspond to the boundaries of surfaces, ellipses to circles viewed obliquely.
- Entities often reveal themselves by a changing relative position to the background as they or the observer move.

Horn [HORN77] relates image intensity distribution to the reflectance characteristics of three-dimensional surfaces, the spatial disposition of a surface and its viewer, and the positions of all light sources. He advocates the use of knowledge about image-formation processes for interpreting the spatial variation of image intensity.

Marr [MARR78] suggests a framework for visual information processing which consists of three major levels of representation: (1) the primal sketch, which makes explicit intensity changes and local two-dimensional geometry, (2) the two-and-a-half-dimensional sketch, which is a viewer-centered representation of depth, orientation, and discontinuities of visible surfaces, and (3) the three-dimensional model, which allows object-centered descriptions of structure and organization. A similar framework is discussed by Ballard et al. [BALL77]. Brooks et al. [BROO78] suggest a more elaborate multilevel structure. The primary representation is in terms of three-dimensional generalized cones for volume elements. A cone is formed by an approximated three-dimensional medial axis and a radius function. A three-dimensional object, then, is a union of generalized cones, whose representation uses three principal graphs: a three-dimensional object graph, a two-dimensional appearance graph, and an observability graph which has both two-dimensional and three-dimensional characteristics. The contents of both the appearance and the observability graphs are in terms of the object graph's contents and may change over the course of recognition and display tasks. The appearance and observability graphs always contain pointers to the object graph and sometimes to each other.

Baer et al. [BAER77] survey ten computer models for three-dimensional objects (also see Requicha [REQU80]). The majority of models reviewed are oriented toward mechanical engineering or architecture. In them, geometric objects are represented in terms of surfaces, edges, and/or vertices. A vertex description is generally employed when the object is to be displayed as a line drawing on a calligraphic monitor or plotter. Surface normals, or the equivalent face coefficients (surfaces), and surface edges (edges) are used when the display approach is raster scan. Some systems store all three types of geometric representations, surfaces, edges, and vertices; others store a minimal amount of data and compute the rest when needed. These are not the only forms of representations used; some systems represent curved surfaces by generic solids (cones, cylinders), for instance, while others use small patches.

Henderson [HEND79] and Nelson et al. [NELS79] describe an approach to fitting surfaces to stereo pairs of aerial photographs. A three-dimensional surface description is developed in real time and compared to a stored world model. In both cases, the application studied is missile guidance.

Electrooptical sensors (EO) are primarily used for military reconnaissance. The main functions of EO image models are to characterize sensor limitations and to delineate the effects of differing weather conditions. Images may be formed by different parts of the electromagnetic spectrum. Under some weather and lighting conditions, forward-looking infrared (FLIR, 8–12 μm) and low light-level television (LLLTV, 0.3–0.7 μm) pictures look very similar to normal visual images. At other times, FLIR and LLLTV produce displays with different characteristics, but different in a predictable manner. FLIR devices are particularly sensitive to the relative changes occurring around sunset as the heating properties of the sun are withdrawn, and the natural patterns of objects' heat emissivity are registered. LLLTV operates in a spectral bandwidth in which each photon contains more energy than FLIR. LLLTVs are capable of operating by faint starlight. LLLTV devices are notoriously oversensitive to some light sources in the night environment, displaying the phenomenon of "blooming," in which small lights enlarge to flood the entire display. Far infrared (FIR, 50–100 μm) devices are in limited use, due to their generally poor resolution.

Microwave radiation (MICRAD, 0.8–1 mm) sensors have relatively low spatial resolution, but high signal-amplitude resolution. Because of the low spatial resolution, object shape takes on a lesser importance in MICRAD imagery. The pixel intensity levels of a MICRAD image bear good witness to the corresponding object's material characteristics, configuration, and surface structure. The high signal resolution of a MICRAD image suggests the use of intensity level and contrast for interpretation. Convergence of evidence is the most common approach employed, with change detection used to search for mobile targets.

Stathacopoulos and Gilmore [STAT77] review 14 models for FLIR and LLLTV imagery. All the models reviewed represent the image by a combination of some of the following descriptors:

(1) target size and position on display,
(2) target angular subtend at the observer's eye,
(3) target background contrast ratio,
(4) target and background luminance,
(5) displayed two-dimensional noise,
(6) background clutter,
(7) resolution of the electrooptical system.

Target and background signal levels are determined by the spectral reflectance and emittance characteristics of the target and background and are affected by both the atmosphere and the transfer function of the electrooptical system. In all the reviewed models, descriptions of targets and backgrounds include only the most basic parameters. Targets are usually represented by rectilinear blocks or are approximated by periodic or bar patterns. For LLLTV, usually target reflectance and luminance are specified; for FLIR, emissivity and temperature difference (with respect to the background) are given. (The atmospheric effect of molecular absorption is of primary importance in the infrared (IR) portion of the spectrum.) All the models use certain approximations to estimate the absorption suffered by radiation in traversing a given amount of water vapor. Some models also consider absorption by CO_2. For LLLTV models, aerosol scattering is a limiting mechanism, except in an extremely clear atmosphere. For IR radiation, aerosol scattering is less crucial, being of importance only in thick haze, fog, or smoke. Not all models include the effects of aerosol scattering; those that do, use a simple formulation applicable only to haze. None of the models treats the effects of atmospheric turbulence. Computations regarding the spectral radiation emitted by targets, the spectral transmission of the atmosphere, and the response of the sensor are usually treated in simplified form.

Schult and Wenthen [SCHU80] use a very theoretical approach to model FLIR scenes. They view a scene as an arrangement of features, a feature as an arrangement of objects, and an object as an arrangement of surfaces. Each surface has a generic material type. A typical scene feature is a building, whose roof may, for example be modeled as layers of paint, steel, pine, plaster, felt, pine, air, and plasterboard. Material characteristics are specified in terms of such properties as conductivity, specific heat, density, absorptivity, and emissivity. Time-invariant thermal properties are evaluated as a function of latitude, longitude, time of day, time of year, and so on. Schult and Wenthen formulate and solve equations describing the thermal properties of each surface within a scene. They then use those surface temperatures in simulating perspective views of the scene.

Moore et al. [MOOR76] discuss techniques for interpreting MICRAD imagery. MICRAD devices use a radiometer to sense the thermal microwave radiation emitted by and reflected from terrain and artifacts. The amount of energy picked up is a function of the emissivity and reflectivity of the observed objects. In the infrared region, radiated power is proportional to the fourth power of thermometric temperature. But at microwave frequencies, power varies almost directly with thermometric temperature and the effects of diurnal changes become small. A target is distinguished from its surroundings by differences in emissivity and reflectivity. Let ΔT denote the contrast between a target and its background. Then

$$\Delta T = E_t T_t - E_b T_b + T_s (E_b - E_t)$$

where E_t, E_b are the emissivity of target and background, respectively, T_t, T_b are the thermometric temperatures of the target and background, and T_s is the radiometric temperature of the sky.

4. CONCLUDING REMARKS

Information in images can usually be described at many levels of abstraction. A description may range from one in terms of semantic attributes of the scene depicted in the image to one that describes only the spatial variation of intensity. Any of these descriptions is an image model which captures only the *relevant* features of the image and leaves others unspecified. The features specified in the model's definition (for a class of images) characterize the unity among the class members—they may have diverse values of other attributes unspecified by the model. Models that involve semantic descriptions are usually called high-level models, and those characterizing the spatial intensity variation are called low-level models.

Our major concern in this paper has been low-level models of homogeneous images or textures. We have surveyed the past work on low-level models under the categories of pixel-based and region-based models. Pixel-based models treat the image as an assignment of values to an array of (random) variables (pixels). The model then specifies a set of rules that defines a legal assignment, the term "legal" requiring that the values of the variables meet certain specified constraints. If the values are restricted only individually, each variable being independent, the result is random noise with a given histogram. For example, if all the variables take gray level values independently chosen from a Gaussian distribution, the result is a Gaussian noise field. Such patterns lack any spatial structure and are, therefore, not interesting for most natural

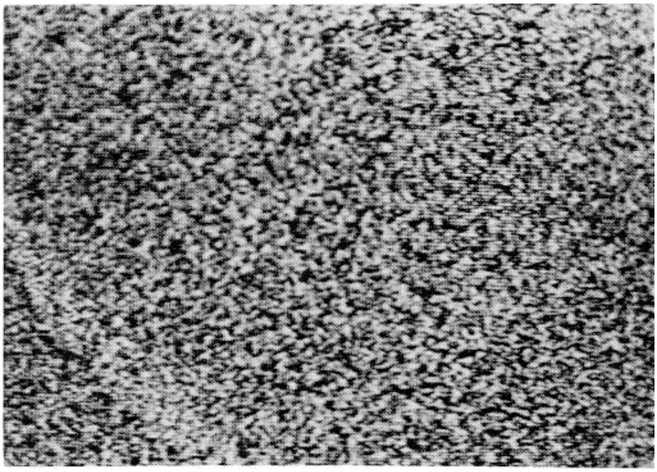


Figure 4. Noise image photographed off a television screen, where the television was not tuned to any channel.

textures (Figure 4). The capability of capturing spatial structure is introduced in the models by specifying joint characteristics of the variables. The choice of properties, and the neighborhood set of pixels which must exhibit those properties, determines the complexity of the model and of the corresponding spatial structure. One-dimensional time series models restrict the neighborhoods to the pixels along a raster scan. This amounts to treating the image as a one-dimensional signal. Even in the more sophisticated time series characterization schemes that capture row periodicity in the raster scan, the image does not receive a truly two-dimensional treatment. This is a drawback and represents a serious limit on the performance of one-dimensional models. Random field models, on the other hand, constrain pixels in *two-dimensional* neighborhoods. Within random field models, many approaches specify relationships among gray levels over the entire image, treating the image as a single neighborhood. In most cases a model of this type is adapted for digital images from one originally intended for Euclidean images (such as one showing the ocean surface, or the earth's relief) and is motivated by certain visually significant, physical (statistical) characteristics of the scenes. Such models may be useful when dealing with images that exhibit known spatial phenomena. However, often it is not possible to relate the spatial variation to a known physical process. In such cases, a variety of schemata (described above as global models) are used to specify the relationships between various image features. We have reviewed models that use the autocorrelation function, variogram, mean, gradient, and their spatial dependence to characterize an image.

Other types of random field models do not specify the nature of the global variation of features and constrain only the values of variables in small neighborhoods. Using local constraints implies the existence of global properties. These models thus are, in a sense, constructive and have a strong "bottom-up" flavor. The local constraints usually involve either a specification of a conditional or a joint probability-like law that relates gray levels of pixels in a given neighorhood. Gray level statistics of pixel pairs have been found to be important for ascertaining human visual perception.

The random-field models, being two dimensional, are more powerful than the one-dimensional models. However, both view an image as an array of pixels. Many natural images, on the other hand, do not exhibit homogeneous pixel properties or gray level neighborhoods, but rather are composed of distinct regions whose geometrical and placement characteristics are uniform throughout the image. This is because many scenes which are sources of image texture consist of finite-sized entities. The shapes and sizes of entities of even a single type may exhibit a statistically describable variation. Thus, unless one deals with scenes consisting of regularly shaped objects, and the resolution is controlled to be sufficiently coarse, the image texture is likely to contain distinct regions, each consisting of samples of a single entity along the sampling grid. For such images, it is natural to base a model upon region properties, rather than upon pixel properties. Region-based models are popular in metallurgy, ecology, astronomy, geology, and forestry, among other disciplines, where they have been motivated by—or seem appropriate for—various physical processes. Some of the so-called structural models in computer image processing have been developed under a similar rationale. However, much more effort has been directed toward developing various forms of pixel-based models than has been toward the region-based models. This may be due to the complexity of the stochastic geometry involved in the latter. Nevertheless, natural images are complex, and complex models are unavoidable.

Many texture studies are basically technique dominated and are based upon texture feature detection and classification schemes, rather than upon a precise image model. We do not discuss these here; see Haralick [HARA78], Mitchell et al. [MITC77], and Thompson [THOM77] for several examples and references.

The idea of an image model has a very broad scope. We may have missed some approaches and may have presented others differently than intended by their authors. Any comments on our treatment are welcome.

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