NON-UNIFORM SAMPLING: A NOVEL APPROACH

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ABSTRACT

In this paper a novel approach to non-uniform sampling is proposed. Two engineering methods are discussed.

Keywords: Sampling, probability, density, partition, Hilbert space.

1. INTRODUTION:

Many natural and man-made signals are analog (continuous)in nature. To represent them in electronic circuits as well as computer memories, we need to discretize the signals. Thus, this process of discretizing an analog signal, called sampling is an important operation and large body of research literature is generated. The instants at which the samples are obtained form a stream of uniform events, which can be depicted graphically as a sampling point process. Characteristic features of the sampled signals to a large extent depend on the patterns of the point process generated and used for sampling [Edi].

When sampling is mentioned in the context of Digital Signal Processing (DSP) , usually it is assumed that the sampling considered is deterministic and periodic. The model of sampling according to which signal samples are considered by time intervals with a constant and known duration is the most popular. Researchers were interested in approximation (of analog signals by discrete time signals) that is as close as possible using periodic sampling. These efforts culminated in Whittaker-Kotel'nikov-Nyquist--Shannon (WKNS) sampling theorem [OWH]. This theorem ensures that the original band-limited signal can be recovered from the samples (periodic) provided the sampling rate is atleast equal to (or greater than) twice the highest frequency to which the original analog signal is bandlimited. If the sampling rate is below the Nyquist rate, reconstruction error, called aliasing occurs. Since a signal is represented by finitely many quantization levels, there is a reconstruction error (of original signal by the samples) due to quantization noise.

Motivation for Non-Uniform Sampling:
As was established relatively long ago, the application of periodic sampling alone does not suffice. The periodic sampling model is not applicable when fluctuations in sampling instants cannot be ignored or when signal samples can be obtained only at irregular or even random time intervals. Also missing samples (from an underlying uniform grid) are a particular case of non-uniform sampling. Studies have indicated that randomness in sampling is not always harmful; sometimes random irregularities in sampling can even be beneficial. These irregularities, if properly introduced, provide, for instance, such an useful effect as the suppression of aliasing**.** And such sampling itself, usually, is considered as non-uniform. Existing non-uniform sampling might be realized either as randomized or pseudo-randomized sampling.
The ultimate goal of various sampling schemes is to decrease the data rate (# of bits to represent a continuous signal) while at the same time providing sufficient amount of accuracy. Our goal in this paper is to approach "optimal sampling" using an innovative approach. Also, in research literature, there are few results devoted to non-uniform sampling of non-band-limited one/two/three/multi dimensional functions. In the following section, we discuss one possible approach to reconstruction of non-bandlimited signals from non-uniform samples.

2. AN INNOVATIVE APPROACH TO NON-UNIFORM SAMPLING:

To describe the approach presented in this section, we need the following information from real analysis.

L^p *Space*: A function f(t) is said

to belong to L^p Space if

$$\int f^p(t)\, dt \, _{< \infty} \quad \ldots\ldots(1)$$

The class of functions in L^2 is said to belong to Hilbert space. Normally, many functions which arise in practical applications belong to Hilbert space.

Innovative Method for Non-Uniform Sampling: Consider a one dimensional function f(x). As in the mathematical discipline, Real Analysis, decompose f(x) in the following manner:

f(x) = g(x) – h(x), (g(x) = f(x) for all "x" such that f(x) > 0) ….(2)

where g(x), h(x) are non-negative functions corresponding to the positive part and negative part of the function f(x).

- In the following, we discuss the non-uniform sampling of g(x). The same approach holds true for the non-uniform sampling of h(x).

- It is very clear that once g(x), h(x) are reconstructed from the samples, the function f(x) can easily be reconstructed.

Consider the function g(x). Let it belong to L^p space for some $1 \le p < \infty$. To be specific, let the function be integrable i.e. the function belongs to L^1 (Clearly all bounded amplitude signals of finite duration are integrable). In the following, we discuss the non-uniform sampling of g(x). Normalize the function in the following manner i.e. Define a new function h(x) such that

$$h(x) = \frac{g(x)}{\int g(x)\, dx}. \quad \ldots\ldots\ldots(3)$$

It is clear that h(x) is a probability density function. We now sample this probability density function. Our goal is to design a sampling scheme such that h(x) can be reconstructed as accurately as possible from the samples. The following approximation approaches are considered.

- The probability density function h(x) has an associated probability distribution function m(x). The problem of sampling boils down to finding a piecewise linear distribution function that approximates (as closely as possible) the distribution function corresponding to h(x). The approximation procedure can be iterative (using a sequence of approximating rectangles). In statistics literature, there are well developed procedures for approximating a density function (or equivalently the distribution function) with respect to some useful/meaningful metric (between the original density and the approximating densities). Those results are invoked in the context of non-uniform sampling.

- The probability density function h(.) is associated with a random variable H. Our goal is to approximate H by a random variable M as closely as possible. From probability literature, Least Mean Square estimation approach is very clear. The minimum mean square estimate of H using M is the conditional expectation of H with respect to M. The formal details of this approach follow from [Pap].

We now propose a new non-uniform sampling approach.
Essential Idea:
Consider the probability density function obtained through the above procedure. For the purposes of Optimal Sampling (resulting in minimum possible reconstruction error), it is logical to include more samples in the region where there is large probability mass and small number of samples in the region where there is small probability mass. The constraint is that the total number of samples is given and fixed.

Note: The problem of approximating a probability density function by a set
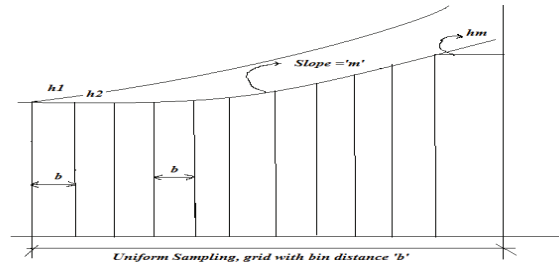of rectangles is well studied in statistics literature. They are effectively transferred to arrive at the notion of optimal sampling.
Two engineering approaches to the problem are summarized in the following. For the sake of simplicity, let the support (domain) of the probability density be finite (bounded). It should be noted that both the engineering approaches are Hybrid Sampling approaches i.e. partly uniform sampling and partly non-uniform sampling approaches.

- FIRST ENGINEERING APPROACH:

COARSE PARTITION DETERMINATION:

1. Consider the RANGE of the probability density function h(x). Based on the minimum as well as maximum possible values, divide the range into finitely many intervals. Using these intervals, find the corresponding COARSE partitions on the domain. i.e. Divide the domain (support) of the probability density function , h(.) into finitely many coarse partitions. Using one of the various numerical integration methods, compute the probability mass in each of the regions of the course partition.

FINE PARTITION DETERMINATION:

2. Decide the "smallest probability mass " in any region of the "fine partition". Using this mass, divide each coarse partition into fine partitions.

- SECOND ENGINEERING APPROACH:

Consider the case where the domain of the function is bounded.
1. Based on the dynamic range of the domain of the function, divide it into equally spaced samples. This constitutes the coarse partition. It corresponds to uniform sampling.

2. Using one of the numerical integration techniques, compute the area in each of the intervals of the coarse partition. Since the function is normalized, these areas correspond to probability values. It is most logical to assign, large number of samples to the interval where there is large probability mass. Thus the fine partition leads to non-uniform sampling of the intervals in the coarse partition.
First let us consider one of the intervals obtained after COARSE sampling. Consider the following diagram illustrating non-uniform sampling of the function restricted to this interval.
Locally the function is approximated by a montone increasing/decreasing function. Suppose the total area of the function restricted to this interval be P_0. Suppose, in this interval, the function is approximated by

Uniform Sampling, grid with bin distance 'b'

rectangles whose base is b_j and height is h_j.

Thus our goal is to find, a finite set of rectangles approximating the function restricted to this interval. Thus, we readily have

$$\sum_{j=1}^{M} h_j b_j = p_0$$

Let $b_j = b$ for j = 1, 2,…., M
Locally, on the interval, the function is approximately utilizing a tangent approximation to the function. Let the slope of the tangent (gradient) line be m. Thus, we readily have that

$$h_1 + m b = h_2$$

$$h_2 + mb = h_2 \qquad \ldots\ldots\ldots(4)$$

$$h_i + mb = h_{i+1} \qquad \text{for } 1 \le i \le (M-1).$$

Furthermore, we readily have

$$b\left[\sum_{i=1}^{M} h_i\right] = p_0.$$

Utilizing the above sequence of equations, we readily have

$$b\left[M h_1 + \frac{M(M-1)}{2} mb \right] = p_0 \quad \ldots\ldots(5)$$

In the above expression, the known quantities are $\{ h_1, m, p_o \}$. We also know the length of the interval, say δ. This length δ is divided into M smaller units. Thus we have

$$b = \frac{\delta}{M} \qquad \ldots\ldots\ldots\ldots(6)$$

Substituting for 'b' (from equation (6)), in

equation (5), we readily have

$$\frac{\delta}{M}\left[M h_1 + \frac{(M-1)}{2} m\delta \right] = p_0 \qquad .$$

The above equation can be explicitly solved for M and the original non-negative function can be sampled.

Remark: An alternative method for computing M (the number of samples in an interval of the course partition) is derived in [Rama] under the condition that the function is strictly increasing over the interval.

3. NON-UNIFORM SAMPLING: INFORMATION-THEORITIC APPROACH:

In the following, we briefly summarize the ideas from source coding in communication theory.

Suppose we have a Discrete Memoryless Source i.e.; the source output random variables are Independent, Identically distributed (IID).

Consider the case where the source output random variables are Discrete i.e.; each random variable X_1 assumes , say L values with probabilities $\{ p_1, p_2, ..., p_L \}$. It is well known from Source Coding Theorem that the average code length is greater than or equal to entropy. Huffman coding achieves the lower bound on average code length. Now we return to the problem of sampling. Consider the coarse partition of the function. It is clear from the second engineering approach that the intervals in the COARSE partition are all of the same length. Let the number of intervals be L and let the corresponding probability masses be $\{ q_1, q_2, ..., q_L \}$.

We have a random variable X that is non-negative and continuous. The support of such a random variable is divided into, say L intervals i.e.;

Probability $\{ 0 < X < x_1 \} = q_1$,

Probability $\{ x_1 \le X < x_2 \} = q_2$, and so on till

Probability$\{ x_{L-1} \le X < x_L \} = q_L$.

It is clear that $\{ q_1, q_2, ..., q_L \}$ are known.

- Also, unlike Huffman coding, we want to assign large number of bits to the interval with large probability mass. The total number of samples is fixed. We are interested in the following question.

- Q: Is there a Lower Bound on the number of bits that can be used to represent/approximate the Continuous Random Variable.

- We utilize the following idea to answer the above question. Consider the continuous random variable on the Partition of values assumed by it. There are L such intervals with probabilities :Prob$\{ x_{(i-1)} \le < X < x_i \} = q_i$.

Define a Discrete Random Variable, Y associated with X i.e. Y assumes only L values.

- We want to assign more symbols to the value of Y with LARGER probability (unlike HUFFMAN CODING).

REFERENCES:
[Edi]   http://www.edi.lv/dasp-web/sec-5.htm
[OWH] A.V. Oppenheim, A. Willsky and H. Nawab, "DiscreteTime Signal Processing ,"Prentice Hall of India
[Pap]   A. Papoulis, "Probability, Random Variables, Stochastic Processes ,"
[Mar] F. Marvasti, "Nonuniform Sampling: Theory and Practice," Kluwer Publish